Taylor Tube Method for Validated IVP*

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Abstract. We recently introduced a novel architecture for the design of validated IVP algorithms. This architecture forms the basis of our complete validated algorithm for IVP. A key subroutine in our algorithm is the Euler Tube: it gave a technique for refining end- and full-enclosures and is also key to deriving a complexity bound of our IVP solver. In this paper, we generalize it to Taylor Tube of degree $p \geq 1$. We stress that our Taylor Tube technique is distinct from the well-known Taylor Models. As expected, higher-degree Taylor tubes improve accuracy. Our experiments further show that it can also lead to an overall speedup when combined with bisection.

Keywords: initial value problem · validated numerics · complete algorithm · Taylor tube method · reachability problem · logarithmic norm

1 Introduction

Consider the initial value problem (IVP) for an autonomous ODE of the form

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}). \tag{1}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is C^k-continuous (for some global constant $k \geq 1$). For $B_0 \subseteq \mathbb{R}^n$ ($n \geq 1$) and $h > 0$, let $\mathtt{IVP}_{\boldsymbol{f}}(B_0, h)$ denote the set of solutions $\boldsymbol{x} : [0, h] \to \mathbb{R}^n$ that satisfy (1) and $\boldsymbol{x}(0) \in B_0$. When the initial set is a singleton, $B_0 = \{\boldsymbol{q}_0\}$, we abbreviate $\mathtt{IVP}_{\boldsymbol{f}}(\{\boldsymbol{q}_0\}, h)$ as $\mathtt{IVP}_{\boldsymbol{f}}(\boldsymbol{q}_0, h)$. We say (B_0, h) is valid (or $\boldsymbol{f}$-valid) if for each $\boldsymbol{p}_0 \in B_0$, some $\boldsymbol{x} \in \mathtt{IVP}_{\boldsymbol{f}}(B_0, h)$ with $\boldsymbol{x}(0) = \boldsymbol{p}_0$ has unique-existence.[1]

For a valid (B_0, h), let

$$\begin{aligned} \mathrm{End}_{\boldsymbol{f}}(B_0, h) &\coloneqq \{\boldsymbol{x}(h) : \boldsymbol{x} \in \mathtt{IVP}_{\boldsymbol{f}}(B_0, h)\}, \\ \mathrm{Image}_{\boldsymbol{f}}(B_0, h) &\coloneqq \{\boldsymbol{x}(t) : \boldsymbol{x} \in \mathtt{IVP}_{\boldsymbol{f}}(B_0, h), t \in [0, h]\}. \end{aligned} \tag{2}$$

We fix $\boldsymbol{f}$ in this paper; so the subscript $\boldsymbol{f}$ may be omitted above. By an **end-closure** and a **full-enclosure** for (B_0, h) we mean any sets E and F that contain $\mathrm{End}(B_0, h)$ and $\mathrm{Image}(B_0, h)$, respectively. Two central problems in the applications of IVP are to compute such end- and full-enclosures. Such enclosure questions are closely related to reachability analysis studied in control theory and its applications [5,13]. The literature on validated algorithms for IVP has a history of over 50 years going back to Moore, and are of course based on interval methods. See the surveys of Corliss [4] and Nedialkov et al. [9].

We formulate the following reachability problem, called the **End Cover Problem**: Given $(\boldsymbol{f}, B_0, h, \varepsilon)$ ($h, \varepsilon > 0$) where B_0 is a box, to compute a finite set $\mathcal{C} = \{B_1, \ldots, B_m\}$ of boxes such that

$$\mathrm{End}_{\boldsymbol{f}}(B_0, h) \subseteq \left(\bigcup \mathcal{C}\right) \subseteq \mathrm{End}_{\boldsymbol{f}}(B_0, h) \oplus [-\varepsilon, \varepsilon]^n \tag{3}$$

where $\bigcup \mathcal{C} = \cup_{i=1}^m B_i$ and $A \oplus B$ denotes the Minkowski sum of Euclidean sets $A, B \subseteq \mathbb{R}^n$. Any $\mathcal{C}$ that satisfies (3) is called an ε**-cover** of $\mathrm{End}_{\boldsymbol{f}}(B_0, h)$.

Recently, we introduced an algorithm [15,16] for this End Cover Problem. It represented the first complete validated IVP algorithm. By "complete" we mean that our algorithm is a halting algorithm. Most validated algorithms are only partially correct[2] in the sense that, the output

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[1] i.e., it exists, and it is unique.

[2] This is a standard terminology in theoretical computer science.

is correct *if it halts*. Halting is especially challenging in IVP problems, partly because the error typically grows exponentially with time. We note some unique features in our solution: (1) We guarantee that the "over-approximation" of $\mathcal{C}$ is within any user-specified ε bound. Rigorous accuracy control generally requires a covering approach rather than a single enclosing box. Even the exact interval hull of the reachable set may be wider than the prescribed tolerance, simply because of the geometry of the set. Thus the End-Cover Problem seeks a finite cover by boxes, instead of one global box enclosure. (2) Our algorithm is fully automatic; it does not require hyper-parameters, such as a minimum step size or tolerance bounds, which are commonly used in such algorithms. (3) In preliminary implementations, our algorithm has proved to be practical. It is reasonable to expect that our algorithm takes a performance hit in order to ensure completeness. Although complete certification usually entails computational overhead, our algorithm often performs better than current validated IVP software [14,3].

Our algorithm [15,16] introduced a novel architecture for IVP algorithms based on the **scaffold** framework. Within this framework, many of our original subroutines could be swapped. One of these subroutines is called **Euler tube**. In this paper, we generalize it into a **Taylor tube** method, based on using the first $p \geq 1$ terms of a Taylor expansion. Call p the **tube degree**. The case of Euler tube corresponds to $p = 1$. In the framework of validated IVP algorithms, our Taylor Tube can be viewed as a technique for implementing stepB in the stepA/stepB iteration (see section 3 below or [15]). Our experiments show that a higher degree for p leads to tighter enclosures (not a surprise) but surprisingly, they can also lead to overall speedup. We remark that such tube techniques are distinct from the well-known Taylor model [1], and they are critical for our ability to prove complexity bounds for our halting algorithm.

2 Taylor Tube

Notations. Most of the following notations are taken from [15,16]. Vector variables such as $\boldsymbol{x} = (x_1, \ldots, x_n)$ or $\boldsymbol{p}$ are in bold font. For $\boldsymbol{p} \in \mathbb{R}^n$ and $r \geq 0$, let $Ball_{\boldsymbol{p}}(r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}-\boldsymbol{p}\|_2 \leq r\}$ denote the Euclidean ball centered at $\boldsymbol{p}$ with radius r. For $r \geq 0$, define the axis-aligned box $Box_{\boldsymbol{p}}(r) := \boldsymbol{p} + [-r, r]^n$. When $\boldsymbol{p} = \boldsymbol{0}$, we simply write $Box(r) := Box_{\boldsymbol{0}}(r) = [-r, r]^n$. For a box $B = \prod_i [\underline{b}_i, \overline{b}_i]$, let $m(B) := \left(\frac{\underline{b}_1+\overline{b}_1}{2}, \cdots, \frac{\underline{b}_n+\overline{b}_n}{2}\right)_i$, $w_{\max}(B) := \max_i(\overline{b}_i - \underline{b}_i)$. For a square matrix $A \in \mathbb{R}^{n\times n}$ and the induced matrix q-norm, the **logarithmic q-norm** is

$$\mu_q(A) \coloneqq \lim_{h\to 0+} \frac{\|I + hA\|_q - 1}{h}.$$

We focus on $q = 2$ and write $\mu(A) = \mu_2(A)$ as the logNorm. For any $B \subseteq \mathbb{R}^n$, define

$$\mu(J_{\boldsymbol{f}}(B)) \coloneqq \sup \{\mu(J_{\boldsymbol{f}}(\boldsymbol{p})) : \boldsymbol{p} \in B\}.$$

Call $\overline{\mu}$ a **logNorm-bound** on B if $\overline{\mu} \geq \mu(J_{\boldsymbol{f}}(B))$. Unlike classical norms, logarithmic norms can be negative. We have this basic result:

Lemma 1. *Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in$ $\mathtt{IVP}(Ball_{\boldsymbol{p}_0}(r), h)$ and assume that $\overline{\mu}$ is a logNorm bound on F where F is any full-enclosure for $(Ball_{\boldsymbol{p}_0}(r), h)$. For all $t \in [0, h]$,*

$$\|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\|_2 \leq \|\boldsymbol{x}_1(0) - \boldsymbol{x}_2(0)\| e^{\overline{\mu}t}. \tag{4}$$

This lemma is just a special case of the logNorm version of the **Fundamental IVP Inequality** ([11,6]).

Tubular region of a trajectory. Fix a trajectory $\boldsymbol{x} = \mathtt{IVP}(\boldsymbol{q}_0, h)$ where $\boldsymbol{x}(0) = \boldsymbol{q}_0$. For any $\delta > 0$, the **δ-tube** of $\boldsymbol{x}$ is the set

$$\mathrm{Tube}_\delta(\boldsymbol{x}) \coloneqq \{(t, \boldsymbol{p}) : 0 \leq t \leq h,\ \|\boldsymbol{p} - \boldsymbol{x}(t)\|_2 \leq \delta\} \quad (\subseteq [0, h] \times \mathbb{R}^n).$$

An arbitrary curve $C : [0,\infty) \to \mathbb{R}^n$ lies in the δ-tube of $\boldsymbol{x}$ if $(t, C(t)) \in \mathrm{Tube}_\delta(\boldsymbol{x})$ for all $t \in [0,h]$. For $p \geq 1$, the Taylor polynomial of $\boldsymbol{f}$ of degree p at any $\boldsymbol{q} \in \mathbb{R}^n$ is defined to be

$$T^p_{\boldsymbol{q}}(t) := \sum_{i=0}^{p} t^i \boldsymbol{f}^{[i]}(\boldsymbol{q}) \tag{5}$$

where [3]

$$\boldsymbol{f}^{[i]}(\boldsymbol{x}) := \begin{cases} \boldsymbol{x} & \text{if } i = 0, \\ \frac{1}{i}\left(J_{\boldsymbol{f}^{[i-1]}} \bullet \boldsymbol{f}\right)(\boldsymbol{x}) & \text{else} \end{cases}$$

where $J_{\boldsymbol{g}}$ denotes the Jacobian of $\boldsymbol{g}$ and $\bullet$ is matrix-vector product. The **Taylor sequence** of $\boldsymbol{q}_0$ with step size $\overline{h} > 0$ is the infinite sequence of points $\boldsymbol{q}_0, \boldsymbol{q}_1, \boldsymbol{q}_2 \ldots$ where $\boldsymbol{q}_{i+1} := T^p_{\boldsymbol{q}_i}(\overline{h})$ for all $i \geq 0$. This sequence defines the piece-wise continuous Taylor curve $T^p_{\boldsymbol{q}_0,\overline{h}}(t)$ of step size $\overline{h}$ where

$$T^p_{\boldsymbol{q}_0,\overline{h}}(t) := T^p_{\boldsymbol{q}_i}(t - i\overline{h}), \qquad (t \geq 0, i = \lfloor t/\overline{h} \rfloor). \tag{6}$$

We could similarly define the δ-tube of the Taylor curve $T^p_{\boldsymbol{q}_0,\overline{h}}(t)$. A natural question is whether this tube contains the true trajectory $\boldsymbol{x}$. The next lemma answers this question in the affirmative, under an explicit upper bound on the Taylor curve with step-size $\overline{h}$. The lemma is a consequence of the fundamental IVP inequality (Lemma 1); its proof can be found in [17].

Lemma 2 (Taylor Tube Step Size Bound). *Let $\boldsymbol{x} = \mathtt{IVP}(\boldsymbol{q}_0, h)$ and B_1 be a full enclosure of $(\boldsymbol{q}_0, h)$. Assume that $\overline{\mu}$ is a logNorm-bound on B_1 and $\overline{M} \geq \|\boldsymbol{f}^{[p+1]}(B_1)\|$. For any $\delta > 0$, if $\overline{h}$ satisfies*

$$\overline{h} \leq h^{taylor}_p(h, \overline{M}, \overline{\mu}, \delta) := \begin{cases} \left(\dfrac{\overline{\mu}\,\delta}{\overline{M}\,(e^{\overline{\mu}h} - 1)}\right)^{1/p} & \text{if } \overline{\mu} > 0, \\ \left(\dfrac{\delta}{\overline{M}h}\right)^{1/p} & \text{if } \overline{\mu} = 0, \\ \min\left\{\left(\dfrac{\overline{\mu}\delta}{2\overline{M}\,(e^{\overline{\mu}h} - 1)}\right)^{1/p}, -\dfrac{1}{\overline{\mu}}\right\}, & \text{if } \overline{\mu} < 0, \end{cases} \tag{7}$$

and if $\mathrm{Image}_{\boldsymbol{f}}(\boldsymbol{q}_i, \overline{h}) \subseteq B_1$ (for all $i = 0, \ldots, \lfloor h/\overline{h} \rfloor$) then the δ-tube of $\boldsymbol{x}$ contains the Taylor curve $T^p_{\boldsymbol{q}_0,\overline{h}}(t)$ of step size $\overline{h}$.

3 Overview of the End-Cover Algorithm

Most validated IVP algorithms are based on two key subroutines which we call[4] `StepA` and `StepB`. Given a box E_0, we may call `StepA`and `StepB` in sequence to transform E_0 into an admissible triple and then admissible quad:

$$E_0 \xrightarrow{\mathtt{StepA}} (E_0, h, F_1) \xrightarrow{\mathtt{StepB}} (E_0, h, F_1, E_1). \tag{8}$$

In general, if (B_0, h) is valid and E, F are end- and full-enclosures for (B_0, h) then (B_0, h, F) (resp., (B_0, h, F, E)) is called an **admissible triple** (resp., **quad**). For $i \geq 1$, we transform $E_{i-1} \longrightarrow (E_{i-1}, h_i, F_i, E_i)$ which is an admissible quad. Note that E_m $(m \geq 1)$ is an end-enclosure for (E_0, t_m) where $t_m = \sum_{i=1}^{m} h_i$. We need admissible triples because global unique-existence of solutions must be built-up incrementally in small time steps which are justified by the classical Picard-Lindelöf theorem.

[3] We may call this the normalized i-th Taylor differential coefficient of $\boldsymbol{f}$, where "differential" here emphasizes that it is not the usual coefficient of polynomial.

[4] `StepA` in [15], has two extra arguments $h, \varepsilon \geq 0$, which is defaulted to $h = \infty, \varepsilon = 0$ here.

Next, we introduce the **scaffold data structure** $\mathcal{S}$ to store such quads. If there are m quads, we use these dot-notations for the various parameters such as

$$\mathcal{S}.m, \quad \mathcal{S}.E_0, \quad \mathcal{S}.t, \quad \mathcal{S}.E_m$$

for the number (m) of quads, the initial box (E_0) of the first quad, the total time (t_m), and the end-enclosure (E_m) at time t_m. Each quad represents a **stage** of $\mathcal{S}$. The goal is to ensure that (a) $\mathcal{S}.t = H$ and (b) $w_{\max}(\mathcal{S}.E_m) < \varepsilon$ where H, ε are the input arguments. If (a) fails, we call a subroutine $\mathcal{S}.\mathtt{Extend}(H)$ that adds a new stage (using `StepA`/`StepB`). If (b) fails, we call $\mathcal{S}.\mathtt{Refine}(\varepsilon)$ to reduce $w_{\max}(\mathcal{S}.E_m)$. However, condition (b) may not be possible without subdividing $\mathcal{S}.E_0$ into subboxes. After subdividing E_0, each subbox $E_0' \subseteq E_0$ becomes the initial box of a new m-stage scaffold that inherits all the quads of $\mathcal{S}$. The basic operation of $\mathcal{S}.\mathtt{Refine}(\varepsilon)$ proceeds in **phases**: each phase refines all m stages (from stage $i = 1, \ldots, m$ in this order). Refining stage i means to convert the ith quad

$$(E_{i-1}, h_i, F_i, E_i) \overset{refine}{\longrightarrow} (E_{i-1}, h_i, F_i', E_i') \tag{9}$$

to a new admissible quad with $F_i' \subseteq F_i$ and $E_i' \subseteq E_i$. If $Q_i = (E_{i-1}, h_i, F_i, E_i)$ then we refine Q_i by generating 2^{ℓ_i} **mini-steps** (with uniform step-size $h_i 2^{-\ell_i}$) between Q_{i-1} and Q_i:

$$Q_{i-1} \longrightarrow Q_i^{(1)} \longrightarrow Q_i^{(2)} \longrightarrow \cdots \longrightarrow Q_i^{(2^{\ell_i}-1)} \longrightarrow Q_i. \tag{10}$$

We provide two subroutines `Bisect` and the `TaylorTube` for doing this refinement: `Bisect` ([15, section 6.3]) simply calls `StepB` twice with half the current mini-step size. `TaylorTube` is similar to `EulerTube` ([15, section 6.3]) except that we use the Taylor sequence $\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots$ and the step size bound h_p^{taylor} in (7). When the mini-step size is smaller than the bound in Lemma 2, we can speed up by using the Taylor tube method described in the next section.

4 Refining End- and Full-Enclosures with Taylor Tubes

We now show how the Taylor Tube method can be combined with `StepB`. One of the best previous methods for `StepB` is the Direct Method in [10,9]:

$$\begin{aligned}\mathtt{Direct}(E_0, h, F_1) &= \underbrace{\sum_{i=0}^{k-1} h^i \boldsymbol{f}^{[i]}(\boldsymbol{m}(E_0))}_{\text{Point}} + \underbrace{h^k \boldsymbol{f}^{[k]}(F_1)}_{\text{Point Error}} + \underbrace{\Big(\sum_{i=0}^{k-1} h^i J_{\boldsymbol{f}^{[i]}}(E_0)\Big) \bullet (E_0 - \boldsymbol{m}(E_0))}_{\text{Range Enclosure}} \\ &= T^{k-1}_{\boldsymbol{m}(E_0)}(h) + \mathtt{PE}(h, F_1) + \mathtt{RE}(h, E_0)\end{aligned} \tag{11}$$

Note that $\mathtt{Direct}(E_0, h, F_1)$ and $\mathtt{Direct}(E_0, [0, h], F_1)$ produce (respectively) an end-enclosure (E_1) and a refinement of the full-enclosure (F_1) for (E_0, h). Our next theorem provides a new method for refining F_1. We state the result here and refer the reader to [17] for the proof.

Theorem 1 (Taylor Tube Enclosures).
Consider an admissible triple (E_0, h, F_1) where $E_0 = Ball_{\boldsymbol{q}_0}(r_0)$.
Let $\delta > 0$, $\overline{h} \le \min\{h, h_p^{taylor}(h, \overline{M}, \overline{\mu}, \delta)\}$, $\overline{\mu}$ is a logNorm-bound on F_1 and $\overline{M} \ge \|\boldsymbol{f}^{[p+1]}(F_1)\|$.
If

$$Image(\boldsymbol{q}_i, \overline{h}) \subseteq F_1 \quad (\text{for all } i = 0, \ldots, \lfloor h/\overline{h} \rfloor) \tag{12}$$

then the following holds:

(a) An end-enclosure for $\mathtt{IVP}(E_0, h)$ *is given by* $Ball_{T^p_{\boldsymbol{q}_0,\overline{h}}(h)}\left(r_0 e^{\overline{\mu} h} + \delta\right)$.

(b) A full-enclosure for $\mathtt{IVP}(E_0, h)$ *is given by* $T^p_{\boldsymbol{q}_0,\overline{h}}([0, h]) \pm [-r, r]^n$, *where* $r = \max_{h' \in [0,h]} \left(r_0 e^{\overline{\mu} h'} + \delta\right) = \delta + \max\{r_0, r_0 e^{\overline{\mu} h}\}$.

We can combine `Direct` with Theorem 1 to define a tighter end-enclosure $E_1(h)$ defined as follows:

$$E_1(h) := \begin{cases} \texttt{Direct}(E_0, h, F_1) \cap \mathrm{Ball}_{T^p_{\boldsymbol{q}_0}(h)}\big(r_0 e^{\overline{\mu} h} + \delta\big), & \text{if } p \neq k-1, \\ T^{k-1}_{\boldsymbol{m}(E_0)}(h) + \big(\texttt{PE}(h, F_1) \cap Box(\delta)\big) + \big(\texttt{RE}(h, E_0) \cap Box(r_0 e^{\overline{\mu} h})\big), & \text{if } p = k-1. \end{cases} \tag{13}$$

Furthermore, $E_1([0,h]) \cap F_1$ gives a tighter full-enclosure than F_1. We compute $E_1([0,h])$ using the formula (13) with the argument $[0,h]$ instead of h.

Our Taylor Tube algorithm can check (12) by using this inclusion

$$T^p_{\boldsymbol{q}_i,\overline{h}}([0,\overline{h}]) + [0,\overline{h}]^{p+1} \boldsymbol{f}^{[p+1]}(F_1) \subseteq F_1. \tag{14}$$

Note that $T^p_{\boldsymbol{q}_i,\overline{h}}([0,\overline{h}])$ and $\boldsymbol{f}^{[p+1]}(F_1)$ have already been computed when computing $E_1([0,\overline{h}])$ and $\overline{M}$.

5 Experiments

We present experimental evidence for the effectiveness of our Taylor tube techniques, using the list of ODE examples in Table 1.

Table 1. List of IVP Problems

Eg*	Name	$\boldsymbol{f}(\boldsymbol{x})$	Parameters	Box B_0	Reference
Eg1	Volterra	$\begin{pmatrix} ax(1-y) \\ -by(1-x) \end{pmatrix}$	$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$	$Box_{(1,3)}(0.1)$	[8], [1, p.13]
Eg2	Van der Pol	$\begin{pmatrix} y \\ -c(1-x^2)y - x \end{pmatrix}$	$c = 1$	$Box_{(-3,3)}(0.1)$	[1, p.2]
Eg3	Asymptote	$\begin{pmatrix} x^2 \\ -y^2 + 7x \end{pmatrix}$	N/A	$Box_{(-1.5,8.5)}(0.01)$	N/A
Eg4	Lorenz	$\begin{pmatrix} \sigma(y-x) \\ x(\rho - z) - y \\ xy - \beta z \end{pmatrix}$	$\begin{pmatrix} \sigma \\ \rho \\ \beta \end{pmatrix} = \begin{pmatrix} 10 \\ 28 \\ 8/3 \end{pmatrix}$	$Box_{(15,15,36)}(0.001)$	[1, p.11]
Eg5	Rössler	$(-y-z, x+ay, b+z(x-c))$	$(a,b,c) = (0.2, 0.2, 5.7)$	$Box_{(1,2,3)}(0.1)$	[12]

5.1 One Step Comparison of Taylor Tube formulas (Table 2)

In Table 2, for each example and for any given admissible (E_0, H, F_1) (which is obtained from `StepA`), we first compute $\overline{h} = \min(H, h^{taylor}(H, \overline{M}, \mu, \delta))$ (see (7)). In this table, we investigate the dependence of $\overline{h}$ on δ: $\overline{h} = \overline{h}(\delta)$. Then we compute the full-enclosure of $\texttt{IVP}(E_0, \overline{h})$ using, respectively, the formulas (11) and (13). We compare the relative effectiveness of the 2 formulas using the ratio

$$\sigma := \frac{vol\big(\texttt{Direct}(E_0, [0,\overline{h}], F_1)\big)}{vol\big(E_1([0,\overline{h}])\big)}, \tag{15}$$

where $vol(\cdot)$ is the volume of a box. Thus $\sigma > 1$ implies an improvement due to Taylor tube. In the table, we choose different values of δ and taylor order k (with $p = k-1$).

From Table 2, we see that σ (last column) is generally > 1 implying the effectiveness of Taylor tube. Moreover, increasing the tube degree causes the step size h^{taylor}_p to increase (see the $\overline{h}$ column). When $\overline{h}$ is capped at the value $\overline{h} = H$, increasing the tube degree can still slightly improve σ, although in this one-step experiment the improvement is often below the two-decimal resolution shown in the table. The significance of this effect is that even small local improvements may accumulate exponentially over many integration steps.

Table 2. Comparison of Full-Enclosures from $Direct(E_0, [0, h], F_1)$ and $E_1([0, h])$.

Eg	E_0	H	F_1	δ	Taylor order k	$\overline{h}$	μ	σ
Volterra	$(1.0, 3.0) \pm (0.1, 0.1)$	0.1	$(0.75, 3.0) \pm (0.36, 0.20)$	0.1	1	0.0144	0.1378	1.00
					3	0.1	0.4386	1.03
					7	0.1	0.4171	1.07
					20	0.1	0.4169	1.07
				0.01	1	0.00201	0.1048	1.09
					3	0.0525	0.2398	1.30
					7	0.1	0.36	1.69
					20	0.1	0.36	1.69
				0.001	1	0.00020	0.1005	1.00
					3	0.0275	0.17	1.23
					7	0.1	1.9	1.85
					20	0.1	0.35	1.85
Van der Pol	$(-3.0, 3.0) \pm (0.1, 0.1)$	0.05	$(-2.9, 2.4) \pm (0.19, 0.71)$	0.1	1	0.0035	0.6314	1.10
					3	0.0481	1.1	1.69
					7	0.05	1.2	1.76
					20	0.05	1.2	1.76
				0.01	1	0.00041	0.60	1.22
					3	0.0239	0.83	1.34
					7	0.05	1.1	2.02
					20	0.05	1.2	2.02
				0.001	1	4.1×10^{-5}	0.60	1.05
					3	0.0114	0.70	1.16
					7	0.05	1.0	2.09
					20	0.05	1.1	2.09
Asymptote	$(-1.50, 8.50)$ $\pm(0.01, 0.01)$	0.04	$(-1.5, 6.8) \pm (0.059, 1.7)$	0.1	1	0.00117	0.1167	1.02
					3	0.0246	2.3	1.42
					7	0.04	3.5	1.97
					20	0.04	3.5	1.97
				0.01	1	0.00012	0.030	1.01
					3	0.0116	1.0	1.22
					7	0.04	3.4	2.14
					20	0.04	3.4	2.15
				0.001	1	1.2×10^{-5}	0.021	1.00
					3	0.0054	0.47	1.10
					7	0.0305	2.8	1.82
					20	0.04	3.3	2.24
Lorenz	$(15.000, 15.000, 36.000)$ $\pm(0.001, 0.001, 0.001)$	0.027	$(15, 13, 38)$ $\pm(0.26, 2.2, 1.8)$	0.1	1	0.000508	2.1	1.000
					3	0.0151	2.9	1.000
					7	0.027	3.3	1.000
					20	0.027	3.3	1.000
				0.01	1	5.3×10^{-5}	2.1	1.000
					3	0.00719	2.5	1.000
					7	0.027	3.3	1.000
					20	0.027	3.3	1.000
				0.001	1	5.4×10^{-6}	2.1	1.000
					3	0.00387	2.3	1.000
					7	0.0248	3.2	1.001
					20	0.027	3.3	1.001
Rössler	$(1.0, 2.0, 3.0)$ $\pm(0.1, 0.1, 0.1)$	0.1	$(0.74, 2.1, 2.3)$ $\pm(0.37, 0.18, 0.85)$	0.1	1	0.00588	0.3155	1.00
					3	0.1	0.71	1.62
					7	0.1	0.70	1.73
					20	0.1	0.70	1.73
				0.01	1	0.00063	0.30	1.00
					3	0.0492	0.48	1.32
					7	0.1	0.65	2.02
					20	0.1	0.65	2.02
				0.001	1	6.3×10^{-5}	0.29	1.00
					3	0.0257	0.39	1.17
					7	0.1	0.64	2.12
					20	0.1	0.64	2.12

5.2 Global Experiment (Table 3)

In contrast to the previous one-step experiments, we now do global experiments using the full algorithm. Table 3 shows the running times of our End-Cover Algorithm on each example using varying tube degrees $(1, 3, 5, 7, 19)$. The order k is fixed at 20. We also record the size $|\mathcal{C}|$ of the end-cover. But in each case, we show two numbers, representing the standard End-Cover Algorithm using `Bisect` (**B**) and its variant (**NoB**) in which the `Bisect` subroutine is turned off. The effect of the tube degree is coupled with other certification mechanisms. When certification fails, the algorithm may reduce the step size or bisect the current initial subbox. Moreover, the Taylor-based enclosure is intersected with the enclosure produced by `TaylorTube`, which is based on a logarithmic-norm version of the fundamental IVP inequality. This bound is often conservative, but not uniformly worse than the Taylor bound, especially for small step sizes. Hence the observed performance depends on the relative sharpness of these bounds on each subproblem. The **NoB** variant removes initial-box subdivision and helps make the influence of the tube degree more visible.

For each example, we add a special row whose tube is degree "19 (**NoT**)". This is our End-Cover Algorithm with `TaylorTube` turned off. Comparing the row for degree 19 with this special row is extremely insightful: we get a triple of timing in seconds. For example, for Volterra, the triple is $(21.7s, 7.06s, 19.3s)$, corresponding to (**T**+**NoB**, **T**+**B**, **NoT**+**B**). Inevitably, the middle value is smallest, showing that the combination of `Bisect` and `TaylorTube` is better than either one alone.

Observe that the use of `Bisect` not only improves the overall efficiency, but also tends to reduce the size of the end-covers (this is seen by comparing the numbers in the **NoB** and **B** columns).

Remark: compared to other tube degrees, the timing for degree 19 is artificially lower since it matches the order $k = 20$ used in `StepB` of (13). This allows the Taylor-tube computation to reuse quantities computed in `StepB` such as the point-error term.

Table 3. Performance of End-Cover algorithm under different tube degrees.

Model	T	ε	**Tube degree** p	Cover size $\lvert\mathcal{C}\rvert$		**Time (s)**	
				NoB	**B**	**NoB**	**B**
Volterra	2	0.01	1	5968	1024	47.5	8.05
			3	5758	1024	44.5	7.82
			5	5758	1024	46.2	10.6
			7	5719	1024	47.3	11.0
			19	2461	1024	21.7	7.06
			19 (**NoT**)	-	1024	-	19.3
Van der Pol	2	0.01	1	13219	4174	75.42	35.58
			3	9196	4195	60.88	35.99
			5	6385	4111	89.94	34.74
			7	4789	4102	99.55	34.48
			19	4732	4098	44.70	34.10
			19 (**NoT**)	-	4096	-	35.28
Asymptote	1	0.01	1	4933	4096	53.05	52.08
			3	4936	4096	56.65	53.64
			5	4930	4096	60.55	53.90
			7	4705	4096	62.13	53.65
			19	4642	4096	62.61	51.14
			19 (**NoT**)	-	4096	-	51.846
Lorenz	1	1.0	1	246	8	10.6	0.293
			3	246	8	14.0	0.296
			5	246	8	16.2	0.302
			7	246	8	18.6	0.300
			19	211	8	36.4	0.274
			19 (**NoT**)	-	8	-	1.75
Rössler	2	1.0	1	8	8	0.146	0.140
			3	8	8	0.165	0.166
			5	8	8	0.188	0.185
			7	8	8	0.199	0.210
			19	8	8	0.138	0.111
			19 (**NoT**)	-	8	-	0.370

6 Conclusion

The introduction of Taylor-tube methods provides a new tool for validated IVP computation. Our experiments show that, within the End-Cover Algorithm, these methods can be effective in producing certified enclosures. A natural direction for future work is to combine our approach with Taylor-model data structures [7,2].

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