

Implementing the biset category of finite groups

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Dedicated to Irene Schreier-Scott and Dana Scott

Abstract. We describe an implementation of the biset category of finite groups as a tower of standard categorical constructions, all of which are implemented in the software project CAP for algorithmic category theory. In particular, we describe the composition of bisets as a composition in a Kleisli category of some biadjunction monad. This composition relies on the universal property of the coequalizer completion of a group viewed as a groupoid on one object. Expressing this universal property offers an elegant categorical interpretation of the Schreier–Sims orbit algorithm. Indeed, the implementation relies on every aspect of the algorithm.

Keywords: biset categories · category of presheaves · coequalizer completion · coproduct completion · ur-algorithms · Schreier–Sims algorithm.

1 Introduction

Throughout this paper G, H, K will denote finite groups. Many of the constructions do not need the finiteness assumption.³

An (H, G) -biset is a set X equipped with a left H -action and a commuting right G -action:

$$h(xg) = (hx)g. \quad (1)$$

The *biset category* **FinBiset** of finite groups [5] has finite groups as objects and isomorphism classes of (H, G) -bisets as morphisms in **FinBiset** (G, H) . Given two bisets ${}_H X_G$ and ${}_K Y_H$ their composition $({}_H X_G, {}_K Y_H) \mapsto {}_K Y_H \circ {}_H X_G$ is defined by

$${}_K Y_H \circ {}_H X_G := {}_K Y \times_H X_G,$$

where $Y \times_H X$ is the set of orbits of the diagonal action of H on $Y \times X$ given by $h(y, x) := (yh^{-1}, hx)$. In Section 2 we introduce the notion of a transitive biset and briefly review the Goursat–Bouc decomposition which decomposes any transitive biset as the composition of elementary transitive bisets.

The goal of this paper is to describe our CAP implementation of the biset category **FinBiset** of finite groups. CAP is an open source software project for algorithmic category theory [8,7,9], which allows the construction of categories as towers of standard categorical constructions. It turns out that **FinBiset** is a full subcategory of the Kleisli category of the biadjunction monad of the finite colimit completion of small categories, where this completion is viewed as a 2-functor, left biadjoint to the functor forgetting colimits. The finite colimit completion of a category is the coequalizer completion of the finite coproduct completion. We explain this in Section 3.

In the case of a group G , viewed as a groupoid on one object, these two completions can be switched due to the orbit theorem. The coequalizer completion of G is nothing but the category of transitive left G -sets. Expressing the universal property of the coequalizer completion amounts to a nontrivial application of the Schreier–Sims orbit algorithm. This turns out to be the crucial step in describing the composition of bisets as the composition in the above-mentioned Kleisli category. This is the content of Section 4.

³ The finiteness assumption makes the sum in the Mackey formula (6) finite.

2 The Goursat-Bouc decomposition

Let ${}_H X_G$ be an (H, G) -biset. Each $g \in G$ defines, due to (1), an H -equivariant automorphism $\chi_g: X \rightarrow X, x \mapsto xg$ of ${}_H X$, inducing a group homomorphism

$$\chi: G \rightarrow \text{Aut}({}_H X), g \mapsto \chi_g, \quad (2)$$

where the composition in $\text{Aut}({}_H X)$ is the precomposition $\chi_g \chi_{g'} = \chi_{gg'}$. In particular, the biset X gives rise to the following right action of G on the set $H \backslash X$ of H -orbits:

$$H \backslash X \times G \rightarrow H \backslash X, (H \cdot x, g) \mapsto H \cdot xg. \quad (3)$$

The biset ${}_H X_G$ is called *transitive* if this action is transitive.

Such a biset can equally be viewed as a left $(H \times G)$ -set, where the action is defined by $(h, g)x := hxg^{-1}$. The biset ${}_H X_G$ is transitive iff this action is transitive. In fact, the transitive (H, G) -bisets are in bijection with the conjugacy classes of subgroups of $H \times G$: A subgroup $D \leq H \times G$ gives rise to a transitive (H, G) -biset $(H \times G)/D$ under the action $h \cdot (x, y)D \cdot g := (hx, g^{-1}y)D$. One can recover D as the stabilizer $D = \text{Stab}_{H \times G}((1, 1)D)$ of $(1, 1)D \in (H \times G)/D$. Conversely, for a transitive (H, G) -biset ${}_H X_G$ the stabilizer

$$D := \text{Stab}_{H \times G}(x) := \{(h, g) \in H \times G \mid hx = xg\}$$

of an $x \in X$ is a subgroup of $H \times G$ and ${}_H X_G \cong (H \times G)/D$ as (H, G) -bisets.

Let Γ be a group, $U \leq \Gamma$ a subgroup of Γ , $N \trianglelefteq \Gamma$ a normal subgroup, and $\varphi: \Gamma' \rightarrow \Gamma$ an isomorphism of groups. Denoting the factor group by $Q := \Gamma/N$, we define the five *elementary* transitive bisets

$$\mathbf{Ind}_U^\Gamma := {}_\Gamma \Gamma_U, \mathbf{Res}_U^\Gamma := {}_U \Gamma_\Gamma, \mathbf{Inf}_Q^\Gamma := {}_\Gamma Q_Q, \mathbf{Def}_Q^\Gamma := {}_Q Q_\Gamma, \mathbf{Iso}(\varphi) := {}_\Gamma \Gamma_{\Gamma'},$$

where maybe the only nonobvious action is the right action of Γ' on Γ via φ . It follows that any transitive (H, G) -biset ${}_H X_G \cong (H \times G)/D$ can be written as the Goursat-Bouc decomposition [5, Section 2.3]:

$${}_H X_G \cong (H \times G)/D \cong \mathbf{Ind}_{P_1}^H \circ \mathbf{Inf}_{P_1/K_1}^{P_1} \circ \mathbf{Iso}(\varphi) \circ \mathbf{Def}_{P_2/K_2}^{P_2} \circ \mathbf{Res}_{P_2}^G, \quad (4)$$

where $P_1 = P_1(D)$, $P_2 = P_2(D)$ denote the projections of D onto H and G , respectively,

$$\begin{aligned} K_1 &= K_1(D) := \{h \in H \mid (h, 1) \in D\} \trianglelefteq P_1, \\ K_2 &= K_2(D) := \{g \in G \mid (1, g) \in D\} \trianglelefteq P_2 \end{aligned}$$

are subnormal subgroups, and $\varphi: P_2/K_2 \xrightarrow{\sim} P_1/K_1$ the isomorphism defined by Goursat's Lemma. One can recover D (and hence the transitive biset) from the two subquotients and the isomorphism via the equality

$$D = \{ (h, g) \in P_1 \times P_2 \mid hK_1 = \varphi(gK_2) \}. \quad (5)$$

The only nontrivial relation among elementary bisets is given by the *Mackey formula*, expressing the (V, U) -biset $\mathbf{Res}_V^\Gamma \circ \mathbf{Ind}_U^\Gamma$ for two subgroups $U, V \leq \Gamma$ as the sum (=coproduct) of transitive (V, U) -bisets

$$\mathbf{Res}_V^\Gamma \circ \mathbf{Ind}_U^\Gamma \cong \sum_{x \in [V \backslash \Gamma / U]} \mathbf{Ind}_{V \cap {}^x U}^V \circ \mathbf{Iso}(\gamma_x) \circ \mathbf{Res}_{V^x \cap U}^U, \quad (6)$$

with isomorphisms $\gamma_x: V^x \cap U \xrightarrow{\sim} V \cap {}^x U$ for each representative x of a double coset in $V \backslash \Gamma / U$.

3 Bisets from a categorical point of view

3.1 Bisets for finite categories

Let **FinCat** denote the 2-category of finite categories, naturally enriched over the elementary topos **FinSet** of finite sets. Let $\mathbf{C}, \mathbf{D}, \mathbf{E} \in \mathbf{FinCat}$ denote finite categories.

Define the category **FinBisetCat** with the same objects as **FinCat** but with functors $\mathbf{C} \rightarrow \mathbf{D}$ replaced by profunctors $X: \mathbf{C} \rightrightarrows \mathbf{D}$, i.e., functors $X: \mathbf{D}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{FinSet}$. For the wide range of applications of bisets for categories see [10]. The composition of profunctors is usually described by a coend-construction, which we do not know how to make algorithmically efficient. Alternatively, one can use the 2-adjunction in **Cat** to rewrite X as a functor

$$X: \mathbf{C} \rightarrow \text{PSh}(\mathbf{D}),$$

where $\text{PSh}(\mathbf{D}) := \text{PSh}(\mathbf{D}, \mathbf{FinSet}) := \text{Func}(\mathbf{D}^{\text{op}}, \mathbf{FinSet})$ denotes the elementary topos of finite-set-valued presheaves on $\mathbf{D}$. Now in order to compose two functors $X: \mathbf{C} \rightarrow \text{PSh}(\mathbf{D})$ and $Y: \mathbf{D} \rightarrow \text{PSh}(\mathbf{E})$ to a functor $XY: \mathbf{C} \rightarrow \text{PSh}(\mathbf{E})$ we first need to lift Y to a (finitely cocontinuous) functor $\hat{Y}: \text{PSh}(\mathbf{D}) \rightarrow \text{PSh}(\mathbf{E})$. We call this composition $XY := X\hat{Y}$ the *Kleisli composition*⁴. The first obvious advantage of this approach is that the 2-category of presheaf categories and cocontinuous functors $\hat{Y}: \text{PSh}(\mathbf{D}) \rightarrow \text{PSh}(\mathbf{E})$ is a strictification of the weak 2-category of categories and profunctors $Y: \mathbf{D} \rightrightarrows \mathbf{E}$.

3.2 The universal property of PSh

A second advantage is the insight that the composition of two profunctors can be reduced to the ur-algorithm⁵ expressing the universal property of the left biadjoint 2-functor $\mathcal{L} = \text{PSh}$: Given a finitely cocomplete category $\mathbf{T}$ and a functor $Y: \mathbf{D} \rightarrow \mathbf{T}$, determine a finitely cocontinuous lift $\hat{Y}: \text{PSh}(\mathbf{D}) \rightarrow \mathbf{T}$. Indeed, the so-called co-Yoneda lemma states that each presheaf $\Omega \in \text{PSh}(\mathbf{D})$ can be “resolved” as the colimit of a diagram of representable presheaves $\text{Hom}_{\mathbf{D}}(-, c)$, while the fully faithful Yoneda functor $\mathcal{Y}: \mathbf{D} \hookrightarrow \text{PSh}(\mathbf{D})$ allows one to consider this diagram as a diagram in $\mathbf{D}$, but where its colimit is taken in $\text{PSh}(\mathbf{D})$ and coincides with Ω . By applying $Y: \mathbf{D} \rightarrow \mathbf{T}$ to the resolving diagram of the presheaf Ω in $\mathbf{D}$ we get a diagram in $\mathbf{T}$, the colimit of which is the desired $\hat{Y}(\Omega)$. The so constructed finitely cocontinuous functor $\hat{Y}: \text{PSh}(\mathbf{D}) \rightarrow \mathbf{T}$ is the extension of Y along $\mathcal{Y}$, expressing the *universal property* of $\text{PSh}(\mathbf{D})$ as a model for the finite colimit completion of $\mathbf{D}$. In the rest of this subsection we will explain how one can realize the finite colimit completion as the coequalizer completion $\text{Coeq}(\sqcup \mathbf{D})$ of the finite coproduct completion $\sqcup \mathbf{D}$ establishing the equivalence

$$\text{PSh}(\mathbf{D}) \simeq \text{Coeq}(\sqcup \mathbf{D}). \quad (7)$$

The algorithmic value of this presheaf-approach to profunctors thus strongly depends on the efficiency of the ur-algorithm realizing the universal property of $\text{PSh}(\mathbf{D})$, which it turn depends on the size of the resolving diagram of the presheaf Ω . A resolving diagram $Q = Q_{\Omega}$ of Ω is formally a quiver with vertices being objects in $\mathbf{D}$ and arrows decorated by morphisms in $\mathbf{D}$.

Such a quiver Q can itself be expressed (in CAP) as a presheaf $Q \in \text{PSh}\left(V \xrightleftharpoons[t]{s} A, \sqcup \mathbf{D}\right)$, i.e., a

coequalizer pair of morphisms $Q(V) \xrightleftharpoons[Q(t)]{Q(s)} Q(A)$ in $\sqcup \mathbf{D}$. By the universal property of the finite

⁴ The similarity with the composition in the Kleisli category is not a coincidence. Since $\text{PSh}(-, \mathbf{Set})$ is a model for the colimit completion of small categories, $\text{PSh}(-, \mathbf{Set})$ is part of a biadjunction and **Biset** is the Kleisli category of the adjunction monad.

⁵ By an *ur-algorithm* we refer in CAP to an algorithm implementing the counit (=evaluation) of a pair $\mathcal{L} \dashv \mathcal{R}$ of biadjoint 2-functors, expressing the universal property of $\mathcal{L}$.

coproduct completion $\sqcup \mathbf{D}$, the Yoneda embedding $\mathcal{Y}$ extends to a full embedding $\sqcup \mathbf{D} \hookrightarrow \mathbf{PSh}(\mathbf{D})$ recovering Ω as the coequalizer in $\mathbf{PSh}(\mathbf{D}) \supset \sqcup \mathbf{D}$:

$$\Omega \xleftarrow{\pi_Q} Q(V) \xrightleftharpoons[Q(t)]{Q(s)} Q(A). \quad (8)$$

A canonical, yet impractical such diagram Q is given by Grothendieck's category of elements. Each object $d \in \mathbf{D}$ appears $|\Omega(d)|$ times as a vertex and each morphism $\varphi: d \rightarrow d'$ appears $|\Omega(d')|$ times as an arrow in the diagram, i.e., $Q(V) = \sqcup_{d \in \mathbf{D}} |\Omega(d)| \otimes d$, $Q(A) = \sqcup_{\varphi: d \rightarrow d' \in \mathbf{D}} |\Omega(d')| \otimes d'$. The description of $Q(s)$ and $Q(t)$ is straightforward. Although the number of arrows in $Q(A)$ can be reduced by only considering generating morphisms of $\mathbf{D}$, the potentially huge number of vertices renders this description algorithmically useless. Finding efficient resolving diagrams Q_Ω to efficiently realize the universal property of $\mathbf{PSh}$ and use it to compose bisets of finite categories are left to future work.

3.3 The category Γ -FinSet as a tower

Let Γ be a finite group, viewed as a groupoid on one object $*$. The presheaf category $\mathbf{PSh}(\Gamma)$ of Γ with values in finite sets is just another encoding of the category Γ -FinSet of finite *left* Γ -sets. Hence, by (7), we get the chain of equivalences:

$$\Gamma\text{-FinSet} \simeq \mathbf{PSh}(\Gamma) \simeq \mathbf{Coeq}(\sqcup \Gamma).$$

An elegant categorical interpretation of the orbit theorem for (finite) groups is that both completions in the categorical tower can be switched (which already fails for the cyclic monoid $\langle e | e^2 = e \rangle$ of order 2), i.e.,

$$\Gamma\text{-FinSet} \simeq \mathbf{Coeq}(\sqcup \Gamma) \simeq \sqcup \mathbf{Coeq}(\Gamma), \quad (9)$$

where the coequalizer completion $\mathbf{Coeq}(\Gamma)$ is (equivalent to) the category of transitive left Γ -sets. This flipped categorical tower will be the basis of the CAP implementation of Γ -FinSet in Section 4.1.

3.4 The biset category of finite groups

From now on, we will denote the full subcategory of **FinBisetCat** on finite groups (viewed as one-object groupoids) by **FinBisetGrp** or simply by **FinBiset**. It is easy to see that this is equivalent to the definition from Section 1, i.e.,

$$\mathbf{FinBiset}(G, H) \cong \{ {}_H X_G \mid {}_H X_G \text{ an } (H, G)\text{-biset} \},$$

where we identify an (H, G) -biset ${}_H X_G$ with a functor

$$X: G \rightarrow H\text{-FinSet} \simeq \mathbf{PSh}(H) \simeq \mathbf{Coeq}(\sqcup H) \simeq \sqcup \mathbf{Coeq}(H). \quad (10)$$

Note that the Kleisli composition $XY \in \mathbf{FinBiset}(G, K)$ of $X \in \mathbf{FinBiset}(G, H)$ and $Y \in \mathbf{FinBiset}(H, K)$ now corresponds to the post-composition ${}_K Y_H \circ {}_H X_G$. The Goursat-Bouc decomposition (4) shows that **FinBiset** is generated (as an $\mathbb{N}$ -linear category) by the elementary transitive bisets, subject to various commutativity relations and the Mackey relation (6).

4 Implementation in CAP

As explained in Section 3.1 and detailed in Figure 1 below, the composition XY of bisets reduces to the Kleisli composition $X\hat{Y}$, where $\hat{Y}: H\text{-FinSet} \rightarrow K\text{-FinSet}$ is the finitely cocontinuous lift of $Y: H \rightarrow K\text{-FinSet}$ along the Yoneda embedding $H \hookrightarrow H\text{-FinSet}$. In Section 4.1 we explain how to construct the categorical tower $\sqcup \mathbf{Coeq}(\Gamma) \simeq \Gamma\text{-FinSet}$ (9) and in Section 4.2 how to compute the intermediate lift $\tilde{Y}: \mathbf{Coeq}(H) \rightarrow \sqcup \mathbf{Coeq}(K)$ (see Figure 1), which is the algorithmically crucial part of the finitely cocontinuous lift $\hat{Y}: \sqcup \mathbf{Coeq}(H) \rightarrow \sqcup \mathbf{Coeq}(K)$.

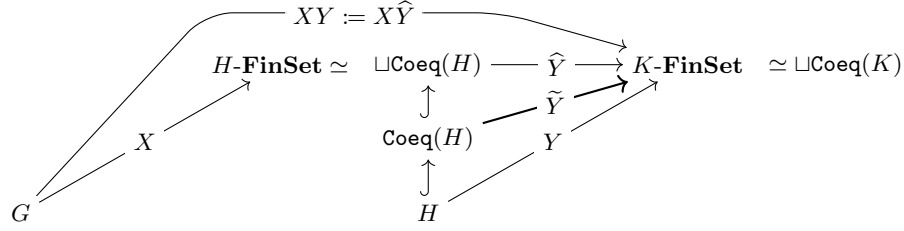


Fig. 1. The Kleisli composition $X\hat{Y}$ computes the composition XY of bisets

4.1 Implementing the category Γ -FinSet as a tower

The coequalizer completion of Γ . A *skeletal* model of the category $\text{Coeq}(\Gamma)$ of transitive *left* Γ -sets can be easily constructed: Let $\{U_c\}_{c=1}^\ell$ be the set of representatives of the ℓ many conjugacy classes of subgroups of Γ with $c \leq d \implies |U_c| \leq |U_d|$, in particular $U_1 = \{1_\Gamma\}$ (the trivial subgroup) and $U_\ell = \Gamma$. Define the skeletal category $\text{Coeq}(\Gamma)$ with objects $c \in \{1, \dots, \ell\}$, corresponding to the *transitive* Γ -sets $\{\Gamma/U_c\}_{c=1}^\ell$. In particular, 1 now stands for the principal Γ -set. A morphism $\gamma: s \rightarrow t$ is given by a group element $\gamma \in \Gamma$ satisfying

$$\gamma^{-1}U_s\gamma =: U_s^\gamma \leq U_t, \quad (11)$$

where two parallel morphisms $\gamma, \gamma': s \rightarrow t$ are considered equal iff $\gamma^{-1}\gamma' \in U_t$. The identity on c is $1_\Gamma: c \rightarrow c$ and the composition $s \xrightarrow{\gamma} c \xrightarrow{\gamma'} t$ is given by the group multiplication $s \xrightarrow{\gamma\gamma'} t$. In $\text{Coeq}(\Gamma)$ all morphisms are epis, and the classes of monos, endomorphisms, and automorphisms coincide. The embedding $\Gamma \hookrightarrow \text{Coeq}(\Gamma)$ sends the unique object $*$ to 1 (corresponding to the principal Γ -set Γ/U_1) and each $\gamma \in \Gamma$ to the automorphism $\gamma: 1 \rightarrow 1$.

Indeed, the category $\mathbf{C} = \text{Coeq}(\Gamma)$ has all coequalizers. However, in what follows we will only need specific coequalizers, namely the coequalizer $\text{coeq}_c(\alpha) = \text{coeq}(\text{id}_c, \alpha_1, \dots, \alpha_k)$ of the identity id_c of an object $c \in \{1, \dots, \ell\}$ and k many automorphisms $\alpha = (\alpha_1, \dots, \alpha_k)$ of c , where $\alpha_r: c \rightarrow c$ and $k \in \mathbb{N}$. To compute $\text{coeq}_c(\alpha)$, first define the subgroup $V := \langle U_c, \alpha_1, \dots, \alpha_k \rangle \leq \Gamma$. Then compute $\gamma \in \Gamma$ such that $V^\gamma = U_d$ for some $d \in \{1, \dots, \ell\}$. It follows that, $\text{coeq}_c(\alpha) = d$ and, using (11) and the definition of V , $\pi = \gamma: c \rightarrow d$ is the projection onto the coequalizer. In the very special case that U_c is self-normalizing it follows that each $\alpha_r = \text{id}_c$, hence $V = U_d = U_c$ and the projection π is given by $1_\Gamma: c \rightarrow c$. Which $U_c \leq \Gamma$ is self-normalizing can be read off the diagonal of the table of marks of Γ . It remains to explain how to compute the colift along the coequalizer: Given an epimorphism $\tau: c \rightarrow t$ with $\tau = \alpha_r\tau$ for all $r = 1, \dots, k$, the colift along the projection $\pi = \gamma: c \rightarrow d$ is simply given by $\gamma^{-1}\tau: d \rightarrow t$.

The universal property of $\text{Coeq}(\Gamma)$ as a coequalizer completion of the one-object groupoid Γ is easy to describe: Given a functor $Y: \Gamma \rightarrow \mathbf{T}$ into a category $\mathbf{T}$ with coequalizers, the lift $\tilde{Y}: \text{Coeq}(\Gamma) \rightarrow \mathbf{T}$ is constructed by the following simple algorithm: Write each object $c \in \text{Coeq}(\Gamma)$ as the coequalizer of the identity $1_\Gamma: 1 \rightarrow 1$ (of the principal Γ -set) and the automorphisms $u_{c1}, \dots, u_{ck_c}: 1 \rightarrow 1$, where $\{u_{c1}, \dots, u_{ck_c}\}$ is any generating set of U_c . Now define $\tilde{Y}(c)$ as the coequalizer in $\mathbf{T}$ of the identity $Y(1_\Gamma) = \text{id}_{Y(*)}$ and the automorphisms $Y(u_{c1}), \dots, Y(u_{ck_c}): Y(*) \rightarrow Y(*)$. Since such a specific coequalizer is central for lifting an action of a group Γ to a functor $\text{Coeq}(\Gamma) \rightarrow \mathbf{T}$ (and then to a functor $\Gamma\text{-FinSet} = \square \text{Coeq}(\Gamma) \rightarrow \mathbf{T}$), we give it a special name in CAP:

$$\text{coeq}_t(\alpha) := \text{CoequalizerOfIdentityAndAutomorphisms}(t, (\alpha_1, \dots, \alpha_k)). \quad (12)$$

In our case $t = Y(*) \in \mathbf{T}$ and $\alpha_r = Y(u_{cr}) \in \text{Aut}_{\mathbf{T}}(t)$ for $r = 1, \dots, k = k_c$.

The finite coproduct completion of a skeletal object-finite category. Let $\mathbf{C}$ be a category with finitely many objects $\{C_1, \dots, C_\ell\}$ (in our application $\mathbf{C} = \text{Coeq}(\Gamma)$ with objects

$\{1, \dots, \ell\} \equiv \{\Gamma/U_1, \dots, \Gamma/U_\ell\}$. If $\mathbf{C}$ is even skeletal, then we can choose a model for the coproduct completion⁶

$$\sqcup \mathbf{C} := \mathbf{FiniteStrictCoproductCompletionOfObjectFiniteCategory}(\mathbf{C}),$$

which is both skeletal and strict: An object $T := \coprod_{c=1}^{\ell} C_c^{\sqcup m_c} \in \sqcup \mathbf{C}$ can be identified with the ℓ -tuple $(m_1, \dots, m_\ell)$ of multiplicities of the objects C_c in the coproduct decomposition of T . In this skeletal data structure the coproduct of two ℓ -tuples is just their componentwise summation.

We will refrain from describing the data structure of a general morphism in $\sqcup \mathbf{C}$ here, since the only morphisms we need in the following subsections are *automorphisms* in $\sqcup \mathbf{C}$, for which we will now describe a simpler data structure: Indeed, being an automorphism of T forces morphisms between the transitive cofactors to remain in the same component. This translates to the isomorphism

$$\mathrm{Aut}_{\sqcup \mathbf{C}}(T) \cong \mathrm{Aut}_{\mathbf{C}}(C_1) \wr S_{m_1} \times \dots \times \mathrm{Aut}_{\mathbf{C}}(C_\ell) \wr S_{m_\ell}, \quad (13)$$

where each element is an ℓ -tuple with c -th entry given by a pair

$$((\alpha_{c1}, \dots, \alpha_{cm_c}), \sigma_c) \in \mathrm{Aut}_{\mathbf{C}}(C_c) \wr S_{m_c} := \mathrm{Aut}_{\mathbf{C}}(C_c)^{\times m_c} \rtimes S_{m_c}.$$

In Section 4.2 we will describe the implementation of `CoequalizerOfIdentityAndAutomorphisms` in $\mathbf{T} := \sqcup \mathbf{C}$ using the Schreier-Sims orbit algorithm.

The category Γ -FinSet as a tower. A *skeletal* model of Γ -FinSet is finally given by the coproduct completion Γ -FinSet $\simeq \sqcup \mathrm{Coeq}(\Gamma)$.

The automorphism group of an object $C_c := \Gamma/U_c \in \mathrm{Coeq}(\Gamma)$ is $\mathrm{Aut}(\Gamma/U_c) \cong N_\Gamma(U_c)/U_c$ by (11), and any automorphism of $\coprod_{c=1}^{\ell} (\Gamma/U_c)^{\sqcup m_c} \in \Gamma$ -FinSet can be modeled according to (13) by an element in the group $\times_{c=1}^{\ell} N_H(U_c) \wr S_{m_c}$, which is again an ℓ -tuple with c -th entry given by a pair $((\nu_{c1}, \dots, \nu_{cm_c}), \sigma_c) \in N_\Gamma(U_c) \wr S_{m_c} \leq \Gamma \wr S_{m_c} := \Gamma^{\times m_c} \rtimes S_{m_c}$.

The finite products in $\sqcup \mathrm{Coeq}(\Gamma)$ can be easily computed using the table of Burnside marks $\{\beta_{\Gamma/U_c}\}_{c=1}^{\ell}$ of Γ with $\beta_{\Gamma/U_c} := \mathrm{Hom}_{\mathrm{Coeq}(\Gamma)}(-, \Gamma/U_c)$. In fact, Γ -FinSet $\simeq \sqcup \mathrm{Coeq}(\Gamma)$ is an elementary topos. We have implemented this tower in the CAP-based package `FinGsetsForCAP`⁷ [2] together with its universal property as a finite colimit completion. This universal property is given by the composition of the ur-algorithm of the coequalizer completion (see Section 4.2) and the ur-algorithm for the finite coproduct completion, the latter being formal. The computational access to this universal property is key for the computation of the Kleisli composition in the biset category, as we have seen in Section 3.1.

4.2 A categorical interpretation of the Schreier-Sims algorithm

In order to implement the Kleisli composition in Figure 1 we need to compute the special coequalizer `CoequalizerOfIdentityAndAutomorphisms` (12) in the category $\mathbf{T} = \sqcup \mathbf{C}$, where $\mathbf{C}$ is a skeletal object-finite category $\{C_1, \dots, C_\ell\}$ equipped with such coequalizers, specialized to $\mathbf{C} = \mathrm{Coeq}(\Gamma)$ for $\Gamma = K$.

It turns out that computing $\mathrm{coeq}_T(\alpha)$ as in (12) for $T \in \mathbf{T} = \sqcup \mathbf{C}$ and $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathrm{Aut}_{\mathbf{T}}(T)^{\times k}$ reduces to the *Schreier-Sims orbit algorithm* and gives the latter an elegant categorical interpretation: Because all morphisms in the above tuple α are automorphisms we can reduce the computation of the coequalizer to each cofactor of $T := \coprod_{c=1}^{\ell} C_c^{\sqcup m_c}$, and then use the functoriality of the coproduct to recover the coequalizer. So we can assume

⁶ The constructor `FiniteStrictCoproductCompletionOfObjectFiniteCategory` was implemented in the CAP-based package `FiniteCocompletions` [3].

⁷ The package contains a primitive implementation of the category $\mathbf{FinSet}\text{-}\Gamma$ of right Γ -sets by the third author. We will replace this primitive implementation by a compilation using `CompilerForCAP` [11,12] of the tower $\sqcup \mathrm{Coeq}(\Gamma^{\mathrm{op}}) \simeq \mathbf{FinSet}\text{-}\Gamma$.

T to be a coproduct $T = C^{\sqcup m}$ of the same object $C \in \mathbf{C}$. For $r = 1, \dots, k$ each $\alpha_r = ((\nu_{r1}, \dots, \nu_{rm}), \sigma_r) \in \text{Aut}_{\mathbf{T}}(T) = \text{Aut}_{\mathbf{C}}(C) \wr S_m$. Define $P := \langle \sigma_1, \dots, \sigma_k \rangle \leq S_m$. Without loss of generality we can assume that P acts transitively on the set $\{1, \dots, m\} \cong_P S \setminus P$ (from the right) with $S = \text{Stab}_P(1)$, otherwise we restrict to P -orbits. Under these assumptions $D := \text{coeq}_T(\alpha) \in \mathbf{C} \subseteq \mathbf{T}$ is a factor object of C in $\mathbf{C}$. In order to compute $D \in \mathbf{C}$ and the m projections $\pi_i: C \rightarrow D$ we invoke the Schreier–Sims orbit algorithm for the transitive permutation group P and obtain m words $t_1 = (), t_2, \dots, t_m$ and $z := mk - m + 1$ words s_j in k indeterminates such that the evaluation $t_i(\sigma_1, \dots, \sigma_k) \in P$ is the transversal of the right coset $St_i(\sigma_1, \dots, \sigma_k) \subseteq P$ corresponding to the point $i \in \{1, \dots, m\}$ and the evaluations $s_j(\sigma_1, \dots, \sigma_k) \in P$ for $j = 1, \dots, z$ generate the stabilizer S . We now replace the σ_r 's with the α_r 's in the evaluations and applying them to $1 \in \{1, \dots, m\}$. Doing this, we first obtain z many automorphisms $\delta_j := 1.s_j(\alpha_1, \dots, \alpha_k) \in \text{Aut}_{\mathbf{C}}(C)$ with $D = \text{coeq}_C(\delta)$ along with the coequalizer projection $\pi: C \rightarrow D$. Finally, we obtain the m projections as $\pi_i := 1.t_i^{-1}(\alpha_1, \dots, \alpha_k)\pi: C \rightarrow D$.

4.3 Action pair as a data structure for a biset

According to (10) a morphism $X \in \mathbf{FinBiset}(G, H)$ corresponding to the (H, G) -biset ${}_H X_G$ is a functor $X: G \rightarrow H\text{-}\mathbf{FinSet} \simeq \sqcup \text{Coeq}(H)$. We now describe the data structure for X . Let G be generated by $\{g_a \mid a = 1, \dots, n\}$:

$$G = \langle g_a \mid a = 1, \dots, n \rangle.$$

The data structure for X is given by an action pair $\chi = ({}_H X, (\chi_a)_{a=1}^n)$ which simply encodes the corresponding group homomorphism $\chi: G \rightarrow \text{Aut}({}_H X)$ from (2), i.e., with $\chi_a := \chi(g_a) \in \text{Aut}({}_H X)$.

According to Section 4.1, the first component of the pair χ is the left H -left ${}_H X \in H\text{-}\mathbf{FinSet}$, which was represented above by an ℓ -tuple $(m_1, \dots, m_\ell)$ of multiplicities. Furthermore, each H -equivariant automorphism χ_a of ${}_H X$ in the second component is given by an ℓ -tuple

$$(\chi_{a,c})_{c=1}^\ell := (((\nu_{a,c1}, \dots, \nu_{a,cm_c}), \sigma_{a,c}))_{c=1}^\ell \in \bigtimes_{c=1}^\ell N_H(U_c) \wr S_{m_c},$$

where $\{U_c\}_{c=1}^\ell$ are representatives of the conjugacy classes of subgroups of H . Note that the action of G described in (3) corresponds to the action of the subgroup $\times_{c=1}^\ell S_{m_c}$ generated by $(\sigma_{a,1}, \dots, \sigma_{a,\ell})$ for $a = 1, \dots, n$.

The additive decomposition of an action pair into an action pair of transitive bisets starts by decomposing an action pair $\chi = ((m_c)_{c=1}^\ell, ((\chi_{a,c})_{c=1}^\ell)_{a=1}^n)$ into its ℓ many components. The c -th component can then be decomposed further by considering the orbits of the permutation group $P_c := \langle \sigma_{a,c} \mid a = 1, \dots, n \rangle_{S_{m_c}}$ on $\{1, \dots, m_c\}$.

The slightly longer arXiv version [4] of this paper describes how to pass from a subgroup $D \leq H \times G$ to the action pair of the corresponding transitive (H, G) -biset $(H \times G)/D$, and vice versa.

5 The relation to the category of generalized morphism in the Abelian group case

One can further restrict **FinBiset** to the full subcategory on finite *Abelian* groups. Mackey's formula (6) states that the composition of two transitive bisets of Abelian groups is at most a multiple of a transitive biset. When ignoring these multiplicities in the composition (e.g., by setting them to 1), the transitive bisets of Abelian groups define a category, which coincides with the category **G(FinAb)** of generalized morphisms of the category **FinAb** of finite Abelian groups. The category of generalized morphisms **G(A)** of a general Abelian category **A** was introduced in [1] and further investigated and implemented in CAP in [6].

While **FinBiset** is enriched over the category of finitely generated free commutative monoids (where the pre-additive structure is a manifestation of the coproduct of I -sets), **G(A)** is enriched over commutative inverse monoids [1].

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