SagePeriods: A SageMath Package for Fast Diagonals of Rational Functions

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Abstract. The field of analytic combinatorics develops effective algorithms to determine asymptotic behaviour of sequences from encodings of their generating functions. After rational and algebraic functions, the next widely studied family of generating functions in combinatorial settings are the *D-finite functions*, which satisfy linear differential equations with polynomial coefficients. One important source of D-finite functions is the class of *diagonals* of multivariate rational functions. We introduce a new SageMath implementation of a reduction-based algorithm due to Lairez, which computes D-finite equations satisfied by multivariate rational diagonals. In addition to bringing this algorithm into the realm of open source computer algebra systems, moving the MAGMA `Periods` package of Lairez to SageMath allows for out-of-the-box compatibility with the existing Sage packages `ore_algebra` and `sage_acsv`, frequently used in algebraic and analytic combinatorics. In particular, we illustrate how combining these packages gives a novel approach to determining *connection coefficients* and exactly computing terms appearing in asymptotics of multivariate rational diagonal sequences.

Keywords: creative telescoping · analytic combinatorics · generating functions.

Let $(f_{\mathbf{i}})_{\mathbf{i}\in\mathbb{N}^d}$ be a d-dimensional complex sequence with *generating function*

$$F(\mathbf{z}) = F(z_1,\dots,z_d) = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}}\mathbf{z}^{\mathbf{i}} = \sum_{i_1,\dots,i_d\geq 0} f_{i_1,\dots,i_d} z_1^{i_1}\cdots z_d^{i_d}.$$

Given a positive integer vector $\mathbf{r}\in\mathbb{N}^d_{>0}$, the $\mathbf{r}$-*diagonal* of $F(\mathbf{z})$ is the univariate generating function

$$(\Delta_{\mathbf{r}}F)(t) = \sum_{n\geq 0} f_{nr_1,\dots,nr_d} t^n$$

for the terms in $(f_{\mathbf{i}})$ whose indices are multiples of $\mathbf{r}$. When $\mathbf{r}=\mathbf{1}$ is the all ones vector we write $(\Delta_{\mathbf{1}}F)(t) = (\Delta F)(t)$ and call ΔF the *main diagonal* of F. Given any vector $\mathbf{r}$ and rational F it is possible to construct another rational function G such that the $\mathbf{r}$-diagonal of F is the main diagonal of G.

A sequence which grows at most exponentially has an analytic generating function at the origin, and the field of *analytic combinatorics* [10,20] uses analytic properties of the generating function to determine asymptotic behaviour of its coefficients. When $d=1$ the classes of *univariate* rational, algebraic, D-finite (satisfying linear differential equations with polynomial coefficients), and D-algebraic (satisfying polynomial differential equations) functions form an increasingly general chain of generating functions appearing frequently in combinatorics. Each element in these classes can be encoded by a finite algebraic or differential equation, but while it is automatic to determine asymptotics for rational and algebraic generating functions appearing in combinatorial problems from these encodings, it is undecidable in general to determine the asymptotic behaviour of a D-algebraic function from an encoding polynomial differential equation and initial conditions. It is an open problem whether asymptotics of sequences with D-finite generating functions can be decided due to the need to exactly compute *connection coefficients*; see Melczer [20, Chapter 2] for more details on these computational aspects of univariate generating functions.

Multivariate functions can be used to track multiple parameters in combinatorial classes, but multivariate diagonals *also* give different representations of univariate sequences. Christol [4] proved that the main diagonal of any multivariate rational function is D-finite, and Lipshitz [19] used a clever counting argument to give an elementary proof that the diagonal of any multivariate D-finite function (which satisfies a linear partial differential equation with respect to each variable individually) is D-finite. The class of rational diagonals is more expressive than it may originally seem – for instance, the diagonal of any d-variate algebraic function can be expressed, in an automatic way, as a diagonal of a $(d+1)$-variate rational function [8,12]. An open conjecture of Christol [5] from 1990 states that any analytic D-finite function $F(z)$ which can be scaled by non-zero $a, b \in \mathbb{Q}$ so that $aF(bz)$ has integer coefficients (including the generating function for any combinatorial class growing at most exponentially) is a rational diagonal. More generally, one can compute diagonals of convergent Laurent expansions – given a rational function F our package computes a D-finite equation satisfied by the diagonals of *all* convergent Laurent expansions of F.

Diagonals are closely linked to a notion of multivariate residues. The *residue* of a suitably convergent multivariate *Laurent series* $F(t, \mathbf{z}) = \sum_{(n,\mathbf{i}) \in \mathbb{N} \times \mathbb{Z}^d} f_{n,\mathbf{i}} t^n \mathbf{z}^{\mathbf{i}}$, with respect to the $\mathbf{z}$ variables, is the univariate series

$$(\text{res } F)(t) = \sum_{n \geq 0} f_{n,-1,-1,\ldots,-1} t^n.$$

Residues of rational functions arise naturally in algebraic geometry since they equal, for suitable values of t and domains of integration Γ, so-called *period integrals*

$$(\text{res } F)(t) = \frac{1}{(2\pi i)^d} \int_{\Gamma} F(t, \mathbf{z}) dz_1 \cdots dz_d.$$

The value of the residue (respectively, the period integral) depends on the domain of convergence of the series (respectively, the domain of integration Γ) however all period integrals with rational integrands over $(d-1)$-cycles are D-finite functions, and for fixed F there exists a D-finite equation satisfied by all periods as Γ ranges over the possible chains of integration.

The problem of computing D-finite equations for rational diagonals is equivalent to the problem of computing D-finite equations for rational period integrals, since

$$(\Delta F)(t) = \text{res } \left(\frac{1}{z_1 \cdots z_d} F\left(\frac{t}{z_1 z_2 \cdots z_d}, z_1, \ldots, z_d \right) \right) \tag{1}$$

and

$$(\text{res } F)(t) = \Delta\left((z_1 \cdots z_d) F\left(t(z_1 z_2 \cdots z_d), z_1, \ldots, z_d \right) \right).$$

1 The SagePeriods Package and Related Software

Algorithms computing D-finite equations of diagonals and residues fall under the purview of *creative telescoping*. A detailed history of creative telescoping, and the different generations of algorithms for this problem, can be found in Chyzak [7]. Prior to the mid-2010s, the main software for computing such D-finite equations were the `Mgfun` package [6] in Maple and the `HolonomicFunctions` package [16] in Mathematica. Lairez [17], building on previous work of Bostan et al. [2], gave the first algorithm for rational period integrals, and thus also diagonals, that is singly exponential in the number of variables. As part of this work, Lairez developed the `Periods` package[3] in MAGMA. Although lacking thorough documentation and requiring a paid MAGMA subscription to use, the package is still the most practical way to compute rational periods and diagonals, having been used to compute hundreds of Picard-Fuchs equations for Calabi-Yau operators [24,18]. In addition, the Sage `ore_algebra` package provides some basic

[3] Available at https://github.com/lairez/periods.

capabilities for creative telescoping [15], the recent `MultivariateCreativeTelescoping.jl` Julia package [3] gives an efficient implementation of general creative telescoping (which handles, but uses less optimized algorithms for, rational functions), and Ramesh and Weiss [23] provide a sage package to compute exact volumes of semi-algebraic convex sets using specialized Picard-Fuchs equations.

In this document we illustrate the main functionality of the `sage_periods` package, bringing Lairez's algorithm to SageMath, adding full documentation, and providing compatibility with the `ore_algebra` [14,21] and `sage_acsv` [13] packages for D-finite equations and asymptotics of multivariate rational functions. The `sage_periods` package can be installed in any recent SageMath installation (preferably version `10.2` or later) by downloading its source code from

https://github.com/ACSVMath/sage_periods

then navigating into the source directory and running

```
sage -pip install .
```

or simply by running the command

```
sage -pip install git+https://github.com/ACSVMath/sage_periods.git
```

from a terminal with access to the internet. The source code repository also contains a Jupyter notebook running through all examples discussed in this paper, and more. Full documentation is available inside SageMath, and on

https://acsvmath.github.io/sage_acsv/

1.1 Illustrating the SagePeriods Package

For the combinatorics user interested in rational diagonals, the most important function is `compute_diagonal_annihilator`, which takes a rational function as an element of the `SymbolicRing` or a multivariate polynomial ring's `FractionField` and returns a representation of a D-finite operator annihilating its (by default) main diagonal. The package detects whether the `ore_algebra` package is available on the user's system: if it is found then the annihilating operator is returned (by default) as an element of the `Univariate Ore algebra in Dt over Univariate Polynomial Ring in t over Rational Field`, while if `ore_algebra` is not found then the operator is returned as an element of the `Ore Polynomial Ring in Dt over Univariate Polynomial Ring in t over Rational Field twisted by d/dt`. Optional arguments allow the user to change the direction vector $\mathbf{r}$, the order of the variables used to convert the diagonal to a residue, and the name of the output variable and differential operator.

Example 1. The computation

```
sage: from sage_periods import compute_diagonal_annihilator
....:     as diagonal
sage: var('w x y z u')
sage: diagonal(1/(1-x-y))
(t - 1/4)*Dt + 1/2
```

implies that the rational diagonal

$$S(t) = \Delta\left(\frac{1}{1-x-y}\right) = \Delta\left(\sum_{i,j\geq 0}\binom{i+j}{i}x^i y^j\right) = \sum_{n\geq 0}\binom{2n}{n} = (1-4t)^{-1/2}$$

satisfies the D-finite equation $(t-1/4)S'(t)+(1/2)S(t)=0$.

The ability to use the functionality of `ore_algebra` with the computed differential operators opens up powerful techniques for combinatorics applications.

Example 2. The *kernel method* is an algebraic technique used (in part) to derive rational diagonal representations for the generating functions of lattice path models [20, Chapter 4]. For instance [20, Example 4.5], the generating function for the number s_n of lattice walks which start at the origin, stay in the nonnegative quadrant $\mathbb{N}^2$, and take n steps in the cardinal directions $\{(\pm 1, 0), (0, \pm 1)\}$ is the main diagonal of $F(x, y, u) = (1+x)(1+y)/(1-uxy(1/x+x+1/y+y))$.

Carrying on from the code executed above, the computation

```
sage: deq = diagonal((1+x)*(1+y)/(1-u*x*y*(1/x+x+1/y+y)))
sage: deq
(t^4 - 1/16*t^2)*D^3 + (8*t^3 + 1/2*t^2 - 3/8*t)*D^2 + (14*t^2 + 7/4*t -
    3/8)*D + 4*t + 3/4
```

implies, after a constant scaling, that the $S(t)$ satisfies the D-finite equation

$$\begin{aligned}(16t^4 - t^2)S'''(t) + (128t^3 + 8t^2 - 6t)S''(t) + (224t^2 + 28t - 6)S'(t)& \\ +(64t + 12)S(t) = 0.&\end{aligned}$$

The coefficient sequence of a D-finite function is always *P-recursive*, meaning it satisfies a linear recurrence with polynomial coefficients. In this case, the computation

```
sage: from ore_algebra import OreAlgebra
sage: rec = deq.to_S(OreAlgebra(QQ['n'], 'Sn'))
sage: rec
(-1/16n^3 - 9/16n^2 - 13/8n - 3/2)Sn^2 + (1/2n^2 + 9/4n + 5/2)Sn + n^3 +
    5n^2 + 8n + 4
```

implies, after a constant scaling, that (s_n) satisfies the recurrence

$$(n^3 - 9n^2 - 26n - 24)s_{n+2} + (8n^2 + 36n + 40)s_{n+1} + (16n^3 + 80n^2 + 128n + 64)s_n = 0.$$

The recurrence can be used to generate terms *very* efficiently. For instance, the command `rec.to_list([1, 2],10001)[-1]` gives s_{10000}, which has around *six thousand* digits, in under a second. After importing `bound_coefficients as BC` from `ore_algebra`, the computation

```
sage: BC(deq, [1, 2], order=1, prec=20)
1.00000*4^n*(([1.2732 +/- 4.39e-5] + [+/- 2.04e-18]*I)*n^(-1) + B
    ([1678.76 +/- 2.19e-3]*n^(-2)*log(n), n >= 5))
```

uses an algorithm from [9] to prove the existence of a (in this case necessarily real) constant $\lambda \in \left[1.2732 - 4.39 \cdot 10^{-5}, 1.2732 + 4.39 \cdot 10^{-5}\right]$ such that

$$s_n - \lambda \frac{4^n}{n} \in \left[\left(1678.76 - 2.19 \cdot 10^{-3}\right) \frac{\log n}{n^2}, \left(1678.76 + 2.19 \cdot 10^{-3}\right) \frac{\log n}{n^2}\right] \tag{2}$$

for all $n \geq 5$. Note that we set a low precision for numerical constants and compute only the leading term in this asymptotic expansion for readability reasons – on a modern laptop one can compute the first 20 terms in the asymptotic expansion of s_n with all constants certified numerically to a thousand digits of accuracy in around a second.

In this case the singular set of $F(x, y, u)$ forms a smooth manifold and all coefficients in the power series expansion of F (not just those on the main diagonal) are non-negative, so after importing `diagonal_asymptotics_combinatorial as diagonal_asm` from the `sage_acsv` package the computation

```
sage: diagonal_asm((1+x)*(1+y)/(1-u*x*y*(1/x+x+1/y+y)))
4/pi*4^n*n^(-1) + O(4^n*n^(-2))
```

proves the leading constant $\lambda = 4/\pi$ in (2).

The constant λ in (2) is related to a *connection coefficient* appearing in the asymptotics of P-recursive sequences. The name reflects the fact that the coefficients which appear connect series expansions of bases of solutions to D-finite equations around different points in the complex plane. Although connection coefficients can be rigorously approximated to thousands of digits using `ore_algebra`, it is an open question whether they can be computed exactly (and, in particular, if one can certify that a coefficient in question is identically zero) which leaves open the decidability of asymptotics for P-recursive sequences. On the other hand, `sage_acsv` computes exact asymptotic expansions, but the difficulty of multivariate techniques means it works under assumptions that may not apply on practical problems. Our `sage_periods` package provides a bridge from multivariate diagonals to D-finite equations, allowing both packages to be used simultaneously to help resolve this tension.

Example 3. The function $F(w, x, y, z) = 1/(1 - w - x - y - z + 27wxyz)$ lies in a family studied by Gillis et al. [11] and has a *cone-point singularity* which prevents `sage_acsv` from directly computing asymptotics of its main diagonal. Baryshnikov et al. [1] give a criterion to rule out so-called *critical points at infinity*, which in this case can be used as a black-box to prove that an asymptotic expansion derived from the D-finite equation satisfied by the diagonal is an *integer multiple* of an asymptotic expansion derived directly from the multivariate rational diagonal representation.

After importing the functions `critical_points`, `get_expansion_terms`, and `compute_asymptotics_at_points as CAP` from `sage_acsv`, the computation

```
sage: F = 1/(1 - (w + x + y + z) + 27*w*x*y*z)
# Get asymptotic expansion from D-finite equation
sage: asm = BC(diagonal(F), [1, -3], order=1)
sage: C1 = list(asm.series_factor().summands)[0].coefficient
# Get asymptotic expansion from rational diagonal
sage: rho = critical_points(F)[1]
sage: asm = get_expansion_terms(CAP(F, [cp]))[0]
sage: C2 = asm.coefficient * asm.pi_factor
# Compare coefficients in both
sage: CIF(C1/C2)
3.00000000000000? + 0.?e-14*I
```

implies that diagonal asymptotics are 3 times the asymptotic expansion with exact coefficients returned by `sage_acsv` (note that there are two dominant asymptotic terms, and an analogous computation shows the integer factor for the second one is also 3) allowing for the exact computation of the connection coefficients. This combination of univariate and multivariate techniques requires a rational diagonal expression for a P-recursive sequence of interest, but is one of the only general approaches currently known for attacking the *connection problem* of determining these coefficients.

Because the package comes with full documentation, we do not spend more time here describing its basic functions. The Jupyter notebook accompanying this article[4] also contains results of running the algorithm on a representative collection of 50 examples from the textbooks [20,22]. The total time to compute annihilators of all these examples was approximately 170 seconds, however two of the examples took over half of the total time and nine of the examples together took up 150 seconds.

2 Summary of Techniques

Our algorithm to compute D-finite equations annihilating rational diagonals simply converts the diagonal to a residue using (1), or a modification for general $\mathbf{r}$-diagonal. To compute an annihilator for the residue we follow the algorithm discussed in Lairez [17, Section 6]. Let $A =$

[4] Available at https://github.com/ACSVMath/sage_periods/blob/main/Example%20Notebook.ipynb

$\mathbb{Q}(t)[\mathbf{z}]$ be polynomials in $z_1,\dots,z_d$ whose coefficients are rational functions in t, and for any $h \in A$ let A_h be the localization of A at h, consisting of all rational functions of the form $g(\mathbf{z})/h(\mathbf{z})^k$ for some $g \in A$ and $k \in \mathbb{N}$. We think of the z_k as variables and t as a parameter.

Given $F(t,\mathbf{z}) = g(t,\mathbf{z})/h(t,\mathbf{z})^k \in A_h$ the goal is to find a differential operator $\mathcal{L}$ in t and ∂_t such that there exist $C_1,\dots,C_d \in A_h$ with

$$\mathcal{L}(F) = \sum_{k=1}^{n} \left(\partial_{z_k} C_k\right)(t,\mathbf{z}). \tag{3}$$

Because $\mathcal{L}$ differentiates only with respect to t, and because the poles of the C_k are contained in the zero set $\mathcal{V}(h)$ of h, we have that

$$\mathcal{L}\left(\int_{\Gamma} F(t,\mathbf{z})d\mathbf{z}\right) = \sum_{k=1}^{n} \int_{\Gamma} \left(\partial_{z_k} C_k\right)(t,\mathbf{z})d\mathbf{z} = 0$$

for suitable closed chains of integration Γ avoiding $\mathcal{V}(h)$, so $\mathcal{L}$ annihilates the residues of F with respect to $\mathbf{z}$. The operator $\mathcal{L}$ in (3) is called a *telescoper* for F, and the C_k are known as *regular certificates* (regular because their pole set is contained in the pole set of F).

If the numerator and denominator of F have degrees bounded by δ then generically the numerators and denominators of the certificates have degrees bounded below by $\delta^{(1-o(1))d^2}$, while a telescoper of minimal degree can be computed in a bit complexity polynomial in δ^d (see Remark 11 and Theorem 12 of Bostan et al. [2]). Computing a telescoper with regular certificate without actually finding the certificate is thus necessary to obtain practically efficient algorithms. Bostan et al. [2] and Lairez [17] compute such telescopers using a *reduction based approach*. First, one homogenizes F to obtain a new rational function $F_{\text{hom}}(t,z_0,z_1,\dots,z_d) = z_0^{-d-1}F(t,z_1/z_0,\dots,z_d/z_0)$ homogeneous of degree $-d-1$. Working projectively helps control the *pole at infinity* during reductions, and the degree is chosen so that a telescoper for F_{hom} is also a telescoper for F.

Write $F_{\text{hom}} = a/f^\ell$ for positive integer ℓ and (necessarily homogeneous) polynomials a and f such that f is square-free; note that a and f may share factors. Reducing a using a Grobner basis of $\text{Jac}(f) = (\partial_{z_0} f,\dots,\partial_{z_d} f)$ gives a relationship $a = r + \sum_{k=0}^d p_k \cdot (\partial_{z_k} f)$ for polynomials r and p_k, and it can be verified that

$$F_{\text{hom}} = \frac{r}{f^\ell} + \frac{\frac{1}{\ell-1}\sum_{k=0}^d \partial_{z_k} p_k}{f^{\ell-1}} + \sum_{k=0}^{d} \partial_{z_k}\left(\frac{-p_k}{(\ell-1)f^{\ell-1}}\right).$$

Ignoring the final sum (which contains pure derivatives that will form an eventual regular certificate) if $\ell > 2$ one can likewise reduce the second term containing $f^{\ell-1}$ in its denominator. Repeating this process expresses F_{hom}, up to a sum of pure derivatives, as a sum of polynomials reduced modulo $\text{Jac}(f)$ over powers of f.

When $\mathcal{V}(f)$ forms a smooth variety in $\mathbb{P}^n$ one has that $\mathbb{Q}(t)[\mathbf{z}]/\text{Jac}(f)$ is finite dimensional, and it can be shown that this reduction lies, up to a sum of pure derivatives, in a finite-dimensional vector space. Continually taking derivatives of F_{hom} with respect to t and performing the same reduction gives additional terms lying in this vector space, so eventually one can compute a non-trivial linear relationship between the partial derivatives which encodes a telescoper. The algorithm in the smooth case is thus theoretically simple and relatively easy to implement using Gröbner basis computations and the division algorithm.

In the non-smooth case, $\mathbb{Q}(t)[\mathbf{z}]/\text{Jac}(f)$ is not finite dimensional and one must reduce modulo *higher order relations*. Lairez [17] develops the advanced differential and algebraic geometry machinery necessary to study and exploit such higher order relations to compute telescopers, and much of the complexity of the software comes from handling the non-smooth case. There are also practical advancements using standard techniques in computer algebra. For instance, instead of working over $\mathbb{Q}(t)$, which is expensive enough to not be practical in many cases, the algorithm picks random primes p_i and interpolation points $t_{i,k}$ and computes relations in

$GF(p_i)$ after evaluating $t = t_{i,k}$. Results can then be interpolated back into $GF(p_i)(t)$, and the differential operators over $\mathbb{Q}(t)$ are reconstructed from those modulo the p_i using the Chinese Remainder Theorem. Unfortunately, because of a lack of tight a priori bounds on the size of the output, the ultimate result is not certified to be correct; a full discussion on this issue and how it can be mitigated is found in Lairez [17, Section 7].

3 Future Work

We plan further improvements to our package, and to its interface with `sage_acsv` and `ore_algebra`. First, although not practical in many cases we will allow the user to work over $\mathbb{Q}(t)$ to have certified output. Second, we plan to implement the *partial certificates* discussed in [17, Section 7.3] which have more reasonable size than the full regular certificates but still allow for certification of a telescoper. We plan to further automate the approach to computing connection coefficients suggested by Example 3, especially after forthcoming work giving an algorithmic approach to detecting critical points at infinity is completed (otherwise the user will have to verify the non-existence of such points independently). Third, the telescopers computed by our package may not have minimal order. As seen in Example 2 it is possible to use a telescoper to get a linear recurrence for the coefficient sequence of interest, and then use this recurrence to generate a large number of terms in the sequence; these terms can be used to *guess* a lower order telescoper (if one exists). The `ore_algebra` package can be used to certify a guessed telescoper $\mathcal{M}$ of a function $f(t)$ from a certified telescoper $\mathcal{L}$ by verifying that $\mathcal{M}$ is a right divisor of $\mathcal{L}$ and annihilates a high-enough order truncation of $f(t)$. Fourth, we hope to incorporate results from an upcoming package implementing the algebraic-to-rational-diagonal reduction of Denef and Lipshitz [8] to allow the user to pass in *algebraic* inputs as well as rational. Finally, we plan to explore possible algorithmic improvements, including multiprocessing of certain stages of the computation, an improved algorithm for Cauchy interpolation, and heuristics (some of which are in the package of Lairez) for picking things such as the variable order used for reductions. Although our (current, unoptimized) implementation has speeds comparable to or exceeding those of the MAGMA package on the examples presented here, we expect our runtime to be worse on larger examples.

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References

1. Baryshnikov, Y., Melczer, S., Pemantle, R.: Stationary points at infinity for analytic combinatorics. Found. Comput. Math. **22**(5), 1631–1664 (2022)
2. Bostan, A., Lairez, P., Salvy, B.: Creative telescoping for rational functions using the Griffiths-Dwork method. In: ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation, pp. 93–100. ACM, New York (2013)
3. Brochet, H., Chyzak, F., Lairez, P.: Faster multivariate integration in D-modules. J. Symbolic Comput. **137**, Paper No. 102569 (2026)
4. Christol, G.: Diagonales de fractions rationnelles et equations différentielles. In: Study group on ultrametric analysis, 10th year: 1982/83, No. 2, pp. Exp. No. 18, 10. Inst. Henri Poincaré, Paris (1984)
5. Christol, G.: Globally bounded solutions of differential equations. In: Analytic number theory (Tokyo, 1988), Lecture Notes in Math., vol. 1434, pp. 45–64. Springer, Berlin (1990)
6. Chyzak, F.: An extension of Zeilberger's fast algorithm to general holonomic functions. Discrete Math. **217**(1-3), 115–134 (2000)

7. Chyzak, F.: The ABC of Creative Telescoping — Algorithms, Bounds, Complexity. Habilitation à diriger des recherches (hdr), University Paris-Sud 11 (Apr 2014)
8. Denef, J., Lipshitz, L.: Algebraic power series and diagonals. J. Number Theory **26**(1), 46–67 (1987)
9. Dong, R., Melczer, S., Mezzarobba, M.: Computing error bounds for asymptotic expansions of regular P-recursive sequences. Math. Comp. **93**(346), 977–1017 (2024)
10. Flajolet, P., Sedgewick, R.: Analytic combinatorics. Cambridge University Press, Cambridge (2009)
11. Gillis, J., Reznick, B., Zeilberger, D.: On elementary methods in positivity theory. SIAM J. Math. Anal. **14**(2), 396–398 (1983)
12. Greenwood, T., Melczer, S., Ruza, T., Wilson, M.C.: Asymptotics of coefficients of algebraic series via embedding into rational series (extended abstract). Sém. Lothar. Combin. **86B**, Art. 30, 12 (2022)
13. Hackl, B., Luo, A., Melczer, S., Schost, E.: A SageMath package for analytic combinatorics in several variables: beyond the smooth case. Sém. Lothar. Combin. **93B**, Art. 159, 12 (2025)
14. Kauers, M., Jaroschek, M., Johansson, F.: Ore polynomials in Sage. In: Computer algebra and polynomials, Lecture Notes in Comput. Sci., vol. 8942, pp. 105–125. Springer, Cham (2015)
15. Kauers, M., Mezzarobba, M.: Multivariate ore polynomials in SageMath. ACM Commun. Comput. Algebra **53**(2), 57–60 (2019)
16. Koutschan, C.: A fast approach to creative telescoping. Math. Comput. Sci. **4**(2-3), 259–266 (2010)
17. Lairez, P.: Computing periods of rational integrals. Math. Comp. **85**(300), 1719–1752 (2016)
18. Lairez, P., Vanhove, P.: Algorithms for minimal Picard-Fuchs operators of Feynman integrals. Lett. Math. Phys. **113**(2), Paper No. 37, 45 (2023)
19. Lipshitz, L.: The diagonal of a D-finite power series is D-finite. J. Algebra **113**(2), 373–378 (1988)
20. Melczer, S.: An Invitation to Analytic Combinatorics: From One to Several Variables. Texts and Monographs in Symbolic Computation, Springer International Publishing (2021)
21. Mezzarobba, M.: Rigorous multiple-precision evaluation of D-finite functions in SageMath. Tech. Rep. 1607.01967, arXiv (2016)
22. Pemantle, R., Wilson, M.C., Melczer, S.: Analytic combinatorics in several variables, Cambridge Studies in Advanced Mathematics, vol. 212. Cambridge University Press, Cambridge, 2nd edn. (2024)
23. Ramesh, L., Weiss, N.: Exact volumes of semi-algebraic convex bodies. Accepted to Proceedings of the 2026 International Symposium on Symbolic and Algebraic Computation (ISSAC 2026) (2026)
24. van Straten, D.: Calabi-Yau operators. In: Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations, Adv. Lect. Math. (ALM), vol. 42, pp. 401–451. Int. Press, Somerville, MA (2018)