Taylor Tubes: A Technique to Improve the Speed and Accuracy of Validated IVP Algorithms[★]

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ABSTRACT

We recently introduced a novel architecture for the design of validated IVP algorithms. This architecture forms the basis of the first complete validated algorithm for IVP. One key component of our previous algorithm is called the **Euler Tube**: this is a technique for refining full enclosures and is also key to deriving a complexity bound for our previous IVP solver. In this paper, we generalize it to **Taylor Tube** of degree $p \geq 1$, where $p = 1$ corresponds to Euler Tube.

As expected, higher-degree Taylor Tubes improve accuracy. But surprisingly, our experiments show that it can also lead to an overall speedup when combined with bisection.

Short Abstract

We are interested in the initial value problem (IVP) for a linear ODE of the form

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}). \tag{1}$$

where $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. For $B_0 \subseteq \mathbb{R}^n$ $(n \geq 1)$ and $h > 0$, let $\mathtt{IVP}_{\boldsymbol{f}}(B_0, h)$ denote the set of solutions $\boldsymbol{x} : [0, h] \to \mathbb{R}^n$ that satisfy (1) and $\boldsymbol{x}(0) \in B_0$. We say (B_0, h) is **valid** if, for each $\boldsymbol{p}_0 \in B_0$, the $\boldsymbol{x} \in \mathtt{IVP}_{\boldsymbol{f}}(B_0)$ with $\boldsymbol{x}(0) = \boldsymbol{p}_0$ has unique-existence.[2] We define the (mathematical) **End Enclosure Problem** for a valid (B_0, h) as the problem of determining the "end slice"

$$\mathrm{End}_{\boldsymbol{f}}(B_0, h) := \left\{ \boldsymbol{x}(h) : \boldsymbol{x} \in \mathtt{IVP}_{\boldsymbol{f}}(B_0, h) \right\}.$$

We are interested in **validated** algorithms. In validated algorithms we must provide explicit interval bounds for any output numerical quantity. The literature on validated algorithms for IVP has a history of over 50 years going back to Moore. See the surveys of Corliss [1] and Nedialkov et al. [2].

We formulate a corresponding (computational) problem called the **End Cover Problem**: Given $(\boldsymbol{f}, B_0, h, \varepsilon)$ where $\varepsilon > 0$, to compute a finite set of boxes $\mathcal{C} = \left\{ B_1, \ldots, B_m \right\}$ such that

$$\mathrm{End}_{\boldsymbol{f}}(B_0, h) \subseteq \left(\bigcup \mathcal{C} \right) \subseteq \mathrm{End}_{\boldsymbol{f}}(B_0, h) \oplus Box(\varepsilon) \tag{2}$$

where $\bigcup \mathcal{C} = \cup_{i=1}^m B_i$, $Box(\varepsilon)$ is the box in $\mathbb{R}^n$ of half-width ε, and $A \oplus B$ denotes Minkowski sum of Euclidean sets $A, B \subseteq \mathbb{R}^n$. We call any $\mathcal{C}$ that satisfies (2) an ε**-cover** of the set $\mathrm{End}_{\boldsymbol{f}}(B_0, h)$.

Recently, we introduced an algorithm [3, 4] for this End Cover Problem. It represented the first **complete** validated IVP algorithm. By "complete" we mean that our algorithm is a halting one. Even stronger, we can provide complexity upper bounds, thanks to the Euler tube technique. In contrast, most validated algorithms are only **partially correct**[3] in the sense that the output is correct *provided it halts*. Halting is especially challenging in IVP problems for various reasons. There are three notable features in our solution:

(N1) We guarantee that the "over-approximation" of $\mathcal{C}$ is within the user-specified ε bound. To our knowledge, no current algorithms attempt to provide such guarantees.

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[2] i.e., it exists, and it is unique.

[3] Using a standard terminology in theoretical computer science.

(N2) We do not require any **hyper-parameters**[4] as inputs to our algorithm. Many numerical algorithms (partly because they are incomplete) allow users to provide hyper-parameters such as minimum-step size, tolerance bounds, etc. By choosing such parameters carefully, users can often coax the algorithm to halt. In any case, incomplete algorithms must fail in one of 3 ways: explicitly, silently (user beware), or loop.

(N3) Our preliminary implementation has proven that our algorithm is practical. One should expect a halting algorithm to take a performance hit (since it takes extra computational effort to ensure completeness). But surprisingly, we often perform better than the current validated algorithm such as VNODE.

In this paper, we build upon our previous work [3, 4]. In particular, we had introduced a general architecture for IVP algorithms based on the **scaffold** framework. Within this framework, most of our computational primitives could be swapped. One such primitive is called the **Euler tube**. We now generalize it into a **Taylor tube** of **degree** $p \geq 1$, defined by using the first $p \geq$ terms of a Taylor expansion. Euler tube corresponds to degree $p = 1$.

Using larger degrees, we fully expected to improve the accuracy of our algorithms; but it is also reasonable to expect increased computational time. But surprisingly, within the limits our experiments, we show that this can also lead to a overall speedup, when Taylor tube is combined with bisection.

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[4] Nevertheless, our implementation has optional hyperparameters that users can provide. Such hyperparameters may speed up a computation, but do not affect the correctness and halting.