Semi-Heuristic Acceleration of Near-Optimal Black Box Polynomial Root-Finders

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Given a positive b, a convex complex domain $\mathbb{S}$, and a dth degree polynomial p, possibly having any root clusters, near-optimal subdivision polynomial root-finders [3, 4, 5] approximate within $1/2^b$ all roots of p in $\mathbb{S}$. The root-finders work even for a *black box polynomial* represented by an oracle (black box subroutine) for its evaluation rather than its coefficients. Such root-finders are much harder to devise but avoid heavy penalty of coefficient swell for shifting and scaling the variable and for root-squaring; moreover, they run faster for polynomials evaluated faster, e.g., shifted sparse or defined by recurrence. Like [3, 4, 5] and unlike [1, 2], our semi-heuristic root-finders allow black box input polynomials with no restriction on their root clusters but accelerate [3, 4, 5] dramatically. Namely, [3, 4, 5] recursively perform σ-soft *exclusion/inclusion (e/i) tests* of discs in the complex plane and also *compress complex discs* well-isolated from external roots; for such a disc D, [3, 4, 5] approximate its sub-disc that shares root set with D and has minimal diameter. σ-soft e/i test of a disc $D(c, \rho) = \{x : |x - c| \leq \rho\}$ stops as soon as one verifies (i) *exclusion* - the disc $D(c, \rho)$ contains no roots of p or (ii) σ*-soft inclusion* - the concentric disc $D(c, \sigma\rho)$, for fixed $\sigma > 1$, contains a root. [3, 4, 5] approximate within $1/2^b$ all m roots in a small neighborhood of $\mathbb{S}$ at the cost of evaluation of p at $O(vm \log(d))$ points with failure probability $O(1/2^v)$ or at $O(d \log(d))$ points deterministically. We decrease these near-optimal upper bounds to $O(\log(d))$ points, with some promise of further decrease to $O(1)$ points, and still certify inclusion in our e/i tests but support correctness of exclusion only empirically. We also accelerate [3, 4, 5] by a factor of over 100 at the disc compression stages - by means of extending e/i test of a complex disc to such a test of a complex ring (annulus).

References

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