FTheoryTools: A computational tool for analysis of singular elliptic fibrations

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OSCAR collaboration

Introduction

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- Nonabelian gauge algebras, matter curves, Yukawa points: **Crepant** resolution, intersection theory
- $\mathrm{U}(1)$ gauge factors and gauge group global structure: Mordell–Weil group
- Discrete gauge factors: Weil–Châtelet group
- Chiral matter: G_4 flux (middle cohomology)
- Vector-like matter: Deligne cohomology, root bundles

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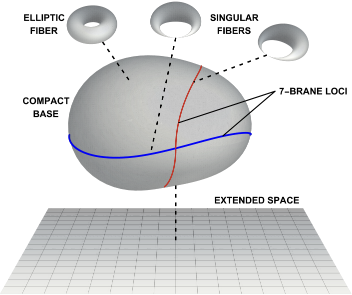
This complexity obstructs progress:

- Imposes a large computational overhead for analyzing models
- Makes it harder for newcomers to enter the field
- Results in duplicated effort

F-theory overview

- Singular elliptically fibered Calabi–Yau n-fold Y:
  - Torus over each point in base B, $\pi : Y \to B$
  - Has a section, $\sigma : B \to Y$ s.t. $\pi \circ \sigma = \mathrm{Id}_B$
  - Complex structure τ encodes Type IIB axiodilaton



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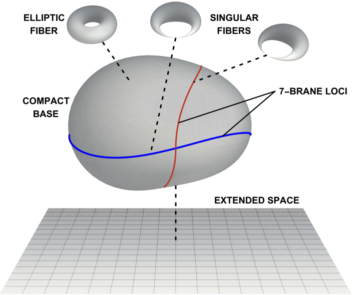

- Described by Weierstrass model: hypersurface

$$y^2 = x^3 + fxz^4 + gz^6$$

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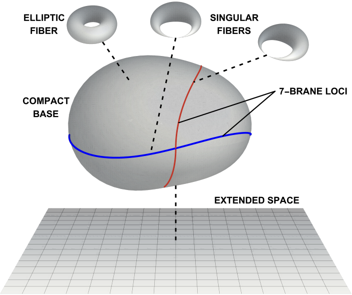

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- Dictionary:
  - Codim.-one singularities (7-branes) $\longleftrightarrow$ nonabelian gauge algebras
  - Codim.-two singularities $\longrightarrow$ massless matter
  - Codim.-three singularities $\longleftrightarrow$ Yukawa couplings
  - Additional (nontorsional) rational sections $\longrightarrow$ $\mathfrak{u}(1)$ gauge algebras
  - Torsional rational sections $\longrightarrow$ global gauge group structure

Kodaira and Refined Tate Fiber Types

Sing. Type	ord(f)	ord(g)	ord(Δ)	Dynkin	$\mathfrak{g}$
I_0	≥ 0	≥ 0	0	—	—
I_1	0	0	1	—	—
II	≥ 1	1	2	—	—
III	1	≥ 2	3	A_1	$\mathfrak{su}(2)$
IV	≥ 2	2	4	A_2	$\mathfrak{sp}(1)$ $\mathfrak{su}(3)$
I_N	0	0	N	A_{N-1}	$\mathfrak{sp}\left(\lfloor \frac{N}{2} \rfloor\right)$ $\mathfrak{su}(N)$
I_0^*	≥ 2	≥ 3	6	D_4	$\mathfrak{g}_2$ $\mathfrak{so}(7)$ $\mathfrak{so}(8)$
I_N^*	2	3	$N+6$	D_{N+4}	$\mathfrak{so}(2N+7)$ $\mathfrak{so}(2N+8)$
IV^*	≥ 3	4	8	E_6	$\mathfrak{f}_4$ $\mathfrak{e}_6$
III^*	3	≥ 5	9	E_7	$\mathfrak{e}_7$
II^*	≥ 4	5	10	E_8	$\mathfrak{e}_8$

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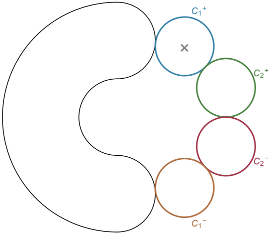
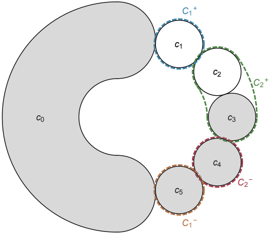
- Can encounter $\mathbb{Q}$-factorial terminal singularities, which *cannot be crepantly resolved*

Fiber Types



$C_1{}^+$
$C_2{}^+$
$C_2{}^-$
$C_1{}^-$



$C_1{}^+$
c_1
c_2
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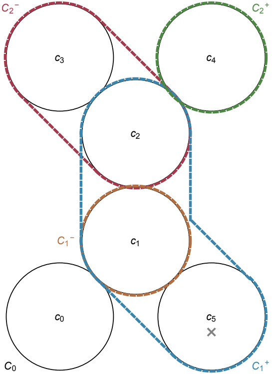
c_0
c_3
c_4
$C_2{}^-$
c_5
$C_1{}^-$



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- Current features:
  - F-theory models over specified or arbitrary bases
  - Multiple model presentations: Weierstrass, Tate, arbitrary hypersurface
  - Model tuning (specialization)
  - (Crepant) Resolution (toric to schemes)
  - Intersection theory and fiber analysis
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- Documentation: https://docs.oscar-system.org/stable/Experimental/FTheoryTools/introduction/
- Tutorial: https://www.oscar-system.org/tutorials/FTheoryTools/

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Example: $\mathrm{U}(1)$-restricted $\mathrm{SU}(5)$ Tate [Krause, Mayrhofer, Weigand '11]

$$y^2 + a_{1,0}xyz + a_{3,2}w^2yz^2 = x^3 + a_{2,1}wx^2z^2 + a_{4,3}w^3xz^4$$

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(Multivariate polynomial ring in 5 variables over QQ, QQMPolyRingElem[a10, a21,
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julia> singular_loci(t)
2-element Vector{Tuple{MPolyIdeal{<:MPolyRingElem}, Tuple{Int64, Int64, Int64},
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 (Ideal with 1 generator, (0, 0, 1), "I_1")
 (Ideal (w), (0, 0, 5), "Split I_5")

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julia> t_res, exceptionals, crepant = blowup_sequence(t, [[6, 7, 5], [2, 3, 1],
       [3, 4], [2, 4]])
(ideal(-b_4_1*e_3*b_2_1*a1*z + ...), [ideal(x, y, w, e_1), ...], true)
```

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`FTheoryTools` includes a database of (families of) models from the F-theory literature:

- Search by arXiv number, DOI, equation number, ...
- Contains as much known data as possible
  - All presentations (Weierstrass, global Tate, hypersurface, ...)
  - Known generating sections
  - Known resolutions
  - Physical data
  - ...

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julia> t = literature_model(arxiv_id = "1109.3454", equation = "3.1")
Global Tate model over a not fully specified base -- SU(5)xU(1) restricted Tate
      model based on arXiv paper 1109.3454 Eq. (3.1)

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Global Tate model over a not fully specified base -- SU(5)xU(1) restricted Tate
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julia> resolve(t, 1)
Partially resolved global Tate model over a not fully specified base --
       SU(5)xU(1) restricted Tate model based on arXiv paper 1109.3454 Eq. (3.1)
```

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To-Do:

- Fluxes (imminent!)
- Mordell–Weil, Weil–Châtelet
- Inclusion of root bundle code `RootCounter`
- CICYs and more general schemes
- Algorithmic **crepant** desingularization?
- Weighted blowups?
- Add many more literature models
- ...