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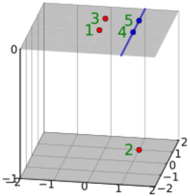

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Exceptional sequences of line bundles on the pentagon

Durham, 23rd July 2024

General notation.

Let X be a smooth projective toric variety over $\mathbb{C}$, e.g. $X_P = \mathbb{P}^2$, $P =$ 

- $\mathbf{D}(X) := \mathbf{D}^{\mathrm{b}}(\mathrm{Coh}(X))$ bounded derived category of X.
- $K(X) := K_0(\mathrm{Coh}(X))$ Grothendieck group of X is of finite rank $N := \mathrm{rk}\, K(X) < \infty$.
- $E_1, \ldots, E_n$ exceptional sequence, $E_i \in \mathrm{Ob}(\mathbf{D}(X))$.

Definition.

A sequence $\mathcal{L}_1, \ldots, \mathcal{L}_n$ of line bundles on X is **exceptional (ES)** iff

$$\mathcal{L}_i - \mathcal{L}_j \in \mathrm{Imm}(X) = \{\mathcal{L} \in \mathrm{Pic}(X) \mid \mathrm{H}^\bullet(X, \mathcal{L}) = 0\} \quad \text{for all } i < j,$$

where $\mathrm{Imm}(X)$ is the **immaculate locus** of X.

Example.

$\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$: $\mathrm{Imm}(\mathbb{P}^n) = \{-n, \ldots, -1\} \Longrightarrow$

$$\mathcal{O}(\ell), \mathcal{O}(\ell+1), \ldots, \mathcal{O}(\ell+n) \text{ is an ES.}$$

Definition.

An exceptional sequence $\mathcal{L}_1, \ldots, \mathcal{L}_n$ on X is

- **maximal (MES)** iff $n = \operatorname{rk} K(X)$.
- **full (FES)** iff $\mathrm{D}(X) = \langle \mathcal{L}_1, \ldots, \mathcal{L}_n \rangle$.

In particular, the implication FES $\Rightarrow$ MES holds.

Example. $\mathcal{O}(\ell), \mathcal{O}(\ell+1), \ldots, \mathcal{O}(\ell+n)$ is full [Beilinson].

MES $\Rightarrow$ FES? No - because of so-called phantom categories.

Kuznetsov's Conjecture (ICM '14).

Let X be a smooth projective variety. If $\mathrm{D}(X) = \langle E_1, \ldots, E_n \rangle$ is generated by an ES, then MES $\Rightarrow$ FES.

Counterexample. [Krah23]
MES $\not\Rightarrow$ FES of line bundles on a rational surface whose derived category is generated by an ES.

When does MES $\Rightarrow$ FES hold?

Currently, it is known to apply to:

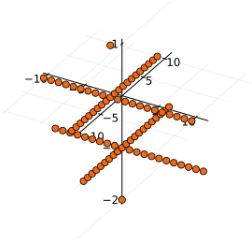
- Smooth projective toric varieties with Picard rank 1 [BEILINSON].
- Smooth projective toric varieties with Picard rank 2 [AW21].
- $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ [AA21].

Next: Does it hold for smooth projective toric varieties with Picard rank $\rho = 3, 4, \ldots$?

Today: concrete calculations for the pentagon $X_P = \mathcal{P}$, $P =$ 

Properties of the pentagon $\mathcal{P}$:

- $\dim \mathcal{P} = 2$.
- $\operatorname{rk} K(\mathcal{P}) = 5$.
- $\operatorname{Pic}(\mathcal{P}) = \mathbb{Z}^3$.
- $\operatorname{Imm}(\mathcal{P})$ is given by:
  - The points $(-1,-1,1)$ and $(0,0,-2)$;
  - The lines $(*,-1,0)$ and $(*,0,-1)$;
  - The lines $(-1,*,0)$ and $(0,*,-1)$.



Construction of MES with Julia:

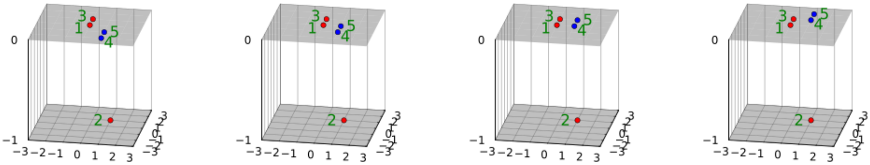
- Set $\mathcal{L}_1 = (0, 0, 0)$.
- Iteratively generate all ES of lengths $2, \ldots, 5$, using condition

$$\mathcal{L}_i - \mathcal{L}_j \in \mathrm{Imm}(\mathcal{P}) \quad \text{for all } i < j.$$

$\rightsquigarrow$ This gives 20 configurations of MES.

Example:

$$\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ p \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ p+1 \\ 0 \end{pmatrix} \right], \quad p \in \mathbb{Z}.$$



Are these configurations full?

Augmentation.

The pentagon $\mathcal{P}$ is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point.
$\rightsquigarrow$ FES on $\mathcal{P}$ can be constructed from those on $\mathbb{P}^1 \times \mathbb{P}^1$:

Theorem. ([LYY18], Thm. 3.6.)

Let $\pi : X \to Y$ be the blow-up of a smooth projective surface Y at a point and E the exceptional divisor of π. A sequence of line bundles,

$$(\mathcal{O}_Y(D_1), \mathcal{O}_Y(D_2), \dots, \mathcal{O}_Y(D_\ell))$$

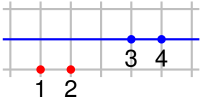
is a FES if and only if, for any $1 \leqslant i \leqslant \ell$,

$$(\mathcal{O}_X(\pi^* D_1 + E), \dots, \mathcal{O}_X(\pi^* D_{i-1} + E), \mathcal{O}_X(\pi^* D_i), \mathcal{O}_X(\pi^* D_i + E), \mathcal{O}_X(\pi^* D_{i+1}), \dots, \mathcal{O}_X(\pi^* D_\ell))$$

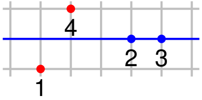
is a FES.

There are four distinct configurations of (normalized) FES on $\mathbb{P}^1 \times \mathbb{P}^1$ [AW21]:

$$\mathrm{ES}_1 = \left[\binom{0}{0}, \binom{1}{0}, \binom{p}{1}, \binom{p+1}{1}\right]$$



$$\mathrm{ES}_2 = \left[\binom{0}{0}, \binom{p}{1}, \binom{p+1}{1}, \binom{1}{2}\right]$$



and their vertical equivalents

$$\mathrm{ES}_3 = \left[\binom{0}{0}, \binom{0}{1}, \binom{1}{p}, \binom{1}{p+1}\right] \quad \text{and} \quad \mathrm{ES}_4 = \left[\binom{0}{0}, \binom{1}{p}, \binom{1}{p+1}, \binom{2}{1}\right],$$

where $p \in \mathbb{Z}$.

We have $E = (1, 1, -1)$ and the pullback is given by

$$\pi^* : \mathbb{Z}^2 \to \mathbb{Z}^3,\ (z_1, z_2) \mapsto (z_2, z_1, 0).$$

Example: For $i = 1$:

$$\begin{aligned} &(\mathcal{O}(D_1),\, \mathcal{O}(D_2),\, \mathcal{O}(D_3),\, \mathcal{O}(D_4))_{\mathbb{P}^1 \times \mathbb{P}^1} \\ \longrightarrow &(\mathcal{O}(\pi^* D_1),\, \mathcal{O}(\pi^* D_1 + E),\, \mathcal{O}(\pi^* D_2),\, \mathcal{O}(\phi^* D_3),\, \mathcal{O}(\pi^* D_4))_{\mathcal{P}} \end{aligned}$$

This gives:

$$\mathrm{ES}_1 = \left[\binom{0}{0}, \binom{1}{0}, \binom{p}{1}, \binom{p+1}{1}\right] \longrightarrow \mathrm{AES}_1^{i=1} = \left[\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\p\\0\end{pmatrix}, \begin{pmatrix}1\\p+1\\0\end{pmatrix}\right].$$

In total, augmentation gives 16 FES.

Helixing.

Let $\mathcal{L}_1, \dots, \mathcal{L}_\ell$ be an ES. The infinite sequence $(\mathcal{L}_i)_{i\in\mathbb{Z}}$ defined by

$$\mathcal{L}_{i+\ell} = \mathcal{L}_i \otimes \omega_X^{-1}, \qquad \omega_X = \text{canonical bundle of } X,$$

is called a **helix of period** ℓ.
In particular, any subsequence $(\mathcal{L}_{k+1}, \dots, \mathcal{L}_{k+\ell})$ of length ℓ is an ES.

Theorem.

If $(\mathcal{L}_1, \dots, \mathcal{L}_\ell)$ is a FES, then any exceptional sequence of the form $(\mathcal{L}_{k+1}, \dots, \mathcal{L}_{k+\ell})$ is full.

Example:

The anticanonical divisor class is $(-1,-1,-1)$, i.e. $\mathcal{L}_{i+5} = \mathcal{L}_i + (1,1,1)$.
The helixings of $\mathrm{AES}_1^{i=1}$ are:

$\mathcal{L}_1$	$\mathcal{L}_2$	$\mathcal{L}_3$	$\mathcal{L}_4$	$\mathcal{L}_5$	$\mathcal{L}_6$	$\mathcal{L}_7$	$\mathcal{L}_8$	$\mathcal{L}_9$	$\mathcal{L}_{10}$
$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\-1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\p\\0\end{pmatrix}$	$\begin{pmatrix}1\\p+1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}2\\2\\0\end{pmatrix}$	$\begin{pmatrix}1\\2\\1\end{pmatrix}$	$\begin{pmatrix}2\\p+1\\1\end{pmatrix}$	$\begin{pmatrix}2\\p+2\\1\end{pmatrix}$
	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}-1\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\p\\1\end{pmatrix}$	$\begin{pmatrix}0\\p+1\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\2\end{pmatrix}$				
		$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\p\\0\end{pmatrix}$	$\begin{pmatrix}1\\p+1\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\\0\end{pmatrix}$			
			$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\p\\1\end{pmatrix}$	$\begin{pmatrix}1\\p+1\\0\end{pmatrix}$	$\begin{pmatrix}0\\p+1\\1\end{pmatrix}$		
				$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\p\\1\end{pmatrix}$	$\begin{pmatrix}1\\p+1\\0\end{pmatrix}$	$\begin{pmatrix}0\\p+1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$	

In summary:

- Helixing produces 80 FES on $\mathcal{P}$.
- There are duplicates: The helixings of $\mathrm{AES}_1^{i=1}$ and $\mathrm{AES}_2^{i=2}$ are identical:

$$\mathrm{ES}_2 = \left[\binom{0}{0}, \binom{p}{1}, \binom{p+1}{1}, \binom{1}{2}\right] \longrightarrow \mathrm{AES}_2^{i=2} = \left[\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\p\\1\end{pmatrix}, \begin{pmatrix}1\\p+1\\0\end{pmatrix}, \begin{pmatrix}0\\p+1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right].$$

- The 20 distinct configurations of FES are exactly the MES coming from Julia.

Conjecture: MES $\Rightarrow$ FES for the pentagon.

Thank you for your attention!

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[Kuz14] A. Kuznetsov, **Semiorthogonal decompositions in algebraic geometry**, arXiv:1404.3143 [math.AG], 2014