Localization in Gromov–Witten theory of toric varieties in a computer algebra system

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Plan of the talk

The goal of this talk is to present the package `ToricAtiyahBott.jl` (see Muratore (2024)). It allows to compute Gromov–Witten invariants of genus 0 of smooth projective toric varieties.

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The plane of the talk is the following.

1. Toric varieties.
2. Gromov–Witten invariants (GW invariants).
3. Examples and possible expansions.

Definition of toric variety

Definition

Let X be a normal complex algebraic variety. We say that X is a toric variety if:

- X contains a torus $T := (\mathbb{C}\backslash\{0\})^n$ as a dense open subset,
- there exists a transitive action of T on itself extending to the whole X.

Definition of toric variety

For each toric variety there is canonical way to define a fan Σ of cones in $\mathbb{Z}^n$, unique modulo isomorphism. Vice versa, fans of cones in $\mathbb{Z}^n$ define a toric variety. So, all properties of a toric variety are coded in its fan.

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Some natural constructions (blow-up, direct product, projectivization of a direct sum of line bundles,...) are closed in the category of toric varieties.

Example

The projective space $\mathbb{P}^n$ contains the torus

$$T = \{[1 : x_1 : \ldots : x_n] \in \mathbb{P}^n / x_i \neq 0\} \cong (\mathbb{C} \backslash \{0\})^n.$$

The standard action of T on $\mathbb{P}^n$ is:

$$\begin{array}{ccccc} T & \times & \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ (t_1, \ldots, t_n) & \times & [x_0 : x_1 : \ldots : x_n] & \mapsto & [x_0 : \frac{x_1}{t_1} : \ldots : \frac{x_n}{t_n}] \end{array}$$

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The $n+1$ points $q_0 = [1 : 0 : \ldots : 0]$, $q_1 = [0 : 1 : 0 : \ldots : 0]$,... are fixed. Moreover, the line through two fixed points is invariant under T.

Example

Every smooth projective toric surface is obtained from either:

- $\mathbb{P}^2$,
- $\mathbb{P}^1 \times \mathbb{P}^1$, or
- $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1})$ for $a \geq 2$,

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Note: A similar package only for GW invariants of $\mathbb{P}^n$ is already available (see, Muratore and Schneider (2022)). The new package extends the applications of the old one.

Properties

From now on, a toric variety X is smooth and projective. It satisfies the following properties:

1. The number of fixed points of the action $T \curvearrowright X$ is finite and greater or equal than two.
2. For each pair of fixed points, there exists at most one T-invariant curve passing through them.
3. Any T-invariant curve is smooth and rational, and passes through exactly two fixed points.

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In particular, any toric variety is also a GKM manifold. The contrary is not true (e.g., Grassmannians other than $\mathbb{P}^n$).
(GKM = Goresky–Kottwitz–MacPherson).

Moduli space of stable maps of genus 0

We denote by $\overline{M}_{0,m}(X,\beta)$ the moduli space of rational stable maps to X of class β and m marks. That is, the moduli of maps $\mu\colon C \to X$ s.t.:

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- C has m marked points $p_i \in C^{reg}$,
- the 1-cycle $\mu_*[C]$ is β,
- if $E \subseteq C$ is an irreducible component mapped to a point, then E contains at least three points among marks and intersection points with other components.

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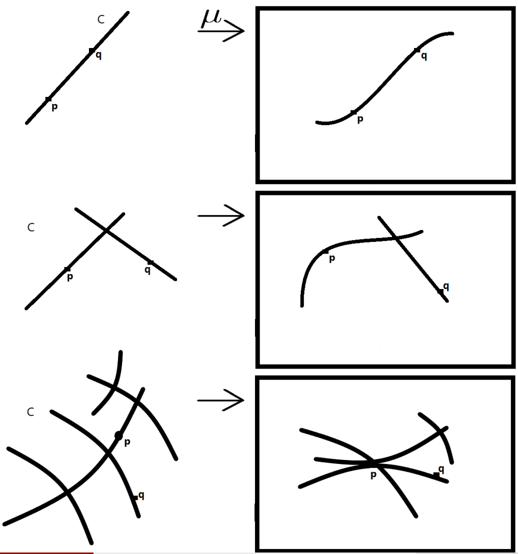
Two stable maps, $(C,\mu,p_1,\dots,p_m)$ and $(C',\mu',p'_1,\dots,p'_m)$, are equivalent if there exists an isomorphism $\phi\colon C \to C'$ such that $\mu = \mu' \circ \phi$ and $\phi(p_i) = p'_i$ for all $i = 1,\dots,m$.

Examples



Moduli space of stable maps of genus 0

Famous examples of this moduli space are:

Example

$\overline{M}_{0,0}(\mathbb{P}^n, 1)$ is (canonically isomorphic to) the Grassmannian of lines in $\mathbb{P}^n$.

Example

$\overline{M}_{0,1}(\mathbb{P}^n, 1)$ is the universal family of the Grassmannian.

Example

$\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is the moduli space of complete conics in $\mathbb{P}^2$.

Gromov–Witten invariants

Definition

We call a Gromov–Witten invariant the degree of the class P:

$$\int_{\overline{M}_{0,m}(X,\beta)} P,$$

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where P is a polynomial in the Chern classes of vector bundles.

The moduli space $\overline{M}_{0,m}(X,\beta)$ and the Gromov–Witten invariants play a crucial role in theoretical physics (string theory) and algebraic geometry. In particular, they are useful in enumerative problems.

Gromov–Witten invariants

As an example, consider the maps $\mathrm{ev}_j \colon \overline{M}_{0,m}(X,\beta) \to X$ such that

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If $V_1,\dots,V_m$ are vector bundles on X of ranks $r_1,\dots,r_m$, then

$$\int_{\overline{M}_{0,m}(X,\beta)} \mathrm{c}_{r_1}(\mathrm{ev}_1^*(V_1))\cdots \mathrm{c}_{r_m}(\mathrm{ev}_m^*(V_m))$$

is a Gromov–Witten invariant.

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Using their properties, there are explicit formualae for computing some GW invariants. Moreover, these formulae reveal hidden properties of X.
The motivation for this project comes from the need to compute GW invariants involving vector bundle and/or varieties where such formulae do not apply. Kontsevich adapted the classical Atiyah–Bott formula to $\overline{M}_{0,m}(X,\beta)$, we implemented it in this package (see Kontsevich 1995).

Kontsevich–Atiyah–Bott formula

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By composition, the action of T lifts to an action of $\overline{M}_{0,m}(X,\beta)$:

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There is a finite number of T-invariant subvarieties $M_\Lambda \subset \overline{M}_{0,m}(X,\beta)$ parameterized by decorated graphs Λ.
A decorated graph is a simple graph with additional structures depending on X, β, m. For example, vertices are colored with the fixed points of X.

Kontsevich–Atiyah–Bott formula

The Kontsevich–Atiyah–Bott formula is

$$\int_{\overline{M}_{0,m}(X,\beta)} P = \sum_{\Lambda} \frac{P^T(\Lambda)}{\mathrm{c}_{\mathrm{top}}^T(N_\Lambda)(\Lambda)},$$

where $P^T(\Lambda)$ is the corresponding equivariant polynomial of P restricted to M_Λ, and N_Λ is the normal bundle of M_Λ.

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where $P^T(\Lambda)$ is the corresponding equivariant polynomial of P restricted to M_Λ, and N_Λ is the normal bundle of M_Λ.
Roughly speaking, this formula permits to compute $\int_{\overline{M}_{0,m}(X,\beta)} P$ as sum of rational polynomials with $\mathbb{Q}$-coefficients and r indeterminates. This sum collapses to a rational number.

Example of Kontsevich–Atiyah–Bott formula

Let $\mathcal{E} \to \overline{M}_{0,0}(\mathbb{P}^3, 1)$ be the Plücker line bundle over the Grassmannian of lines in $\mathbb{P}^3$.

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Thus

$$\int_{\overline{M}_{0,0}(\mathbb{P}^3,1)} \mathrm{c}_1(\mathcal{E})^4$$

is equal to the number of lines meeting four general lines in $\mathbb{P}^3$.

Example of Kontsevich–Atiyah–Bott formula

If q_0, q_1, q_2, q_3 are the fixed points of the T-action on $\mathbb{P}^3$, the decorated graphs are:

$$\Lambda = q_i \text{ ——— } q_j \qquad 0 \leq i < j \leq 3.$$

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Moreover, we can prove that

$$\mathrm{c}_1(\mathcal{E})^T(\Lambda) = \lambda_i + \lambda_j,$$

Example of Kontsevich–Atiyah–Bott formula

Thus, applying the KAB formula we have:

$$\begin{aligned}\int_{\overline{M}_{0,0}(\mathbb{P}^3,1)} \mathrm{c}_1(\mathcal{E})^4 &= \sum_{\Lambda} \frac{(\mathrm{c}_1(\mathcal{E})^T(\Lambda))^4}{\mathrm{c}_{\mathrm{top}}^T(N_\Lambda)(\Lambda)} \\ &= \sum_{0\le i<j\le 3} \frac{(\lambda_i+\lambda_j)^4}{\prod_{k\neq i,j}(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)} \\ &= 2.\end{aligned}$$

Example of Kontsevich–Atiyah–Bott formula

Thus, applying the KAB formula we have:

$$\int_{\overline{M}_{0,0}(\mathbb{P}^3,1)} \mathrm{c}_1(\mathcal{E})^4 = \sum_{\Lambda} \frac{(\mathrm{c}_1(\mathcal{E})^T(\Lambda))^4}{\mathrm{c}^T_{\mathrm{top}}(N_\Lambda)(\Lambda)}$$

$$= \sum_{0\leq i<j\leq 3} \frac{(\lambda_i+\lambda_j)^4}{\prod_{k\neq i,j}(\lambda_i-\lambda_k)(\lambda_j-\lambda_k)}$$

$$= 2.$$

This package performs this computation automatically.

The Package

We used OSCAR, in particular for computing the followings:

- cohomology ring of X,
- generators of the nef cone of X,
- cohomology class of any curve passing through two fixed points.

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It could be possible to extend the package to GKM manifolds, provided that Oscar supports the above tasks. The invariants of those manifolds are very difficult to compute.

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