Chebyshev varieties

[arXiv:2401.12140]

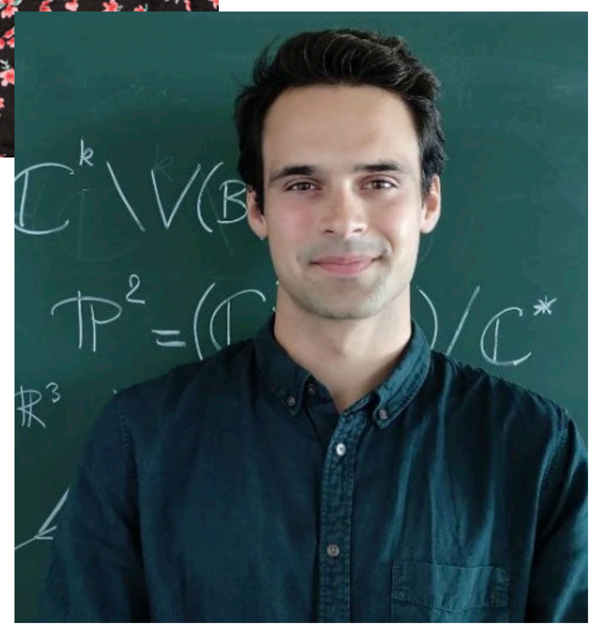
Chiara Meroni

ETH zürich

Institute for Theoretical Studies

Joint work with Zaïneb Bel-Afia

and Simon Telen



International Congress on Mathematical Software 2024

Classical Algebraic Geometry & Modern Computer Algebra: Innovative Software Design and its Applications

Toric geometry

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System of (nonlinear) polynomial equations on $\mathbb{R}^n$ (or $\mathbb{C}^n$) → System of linear equations on an algebraic variety

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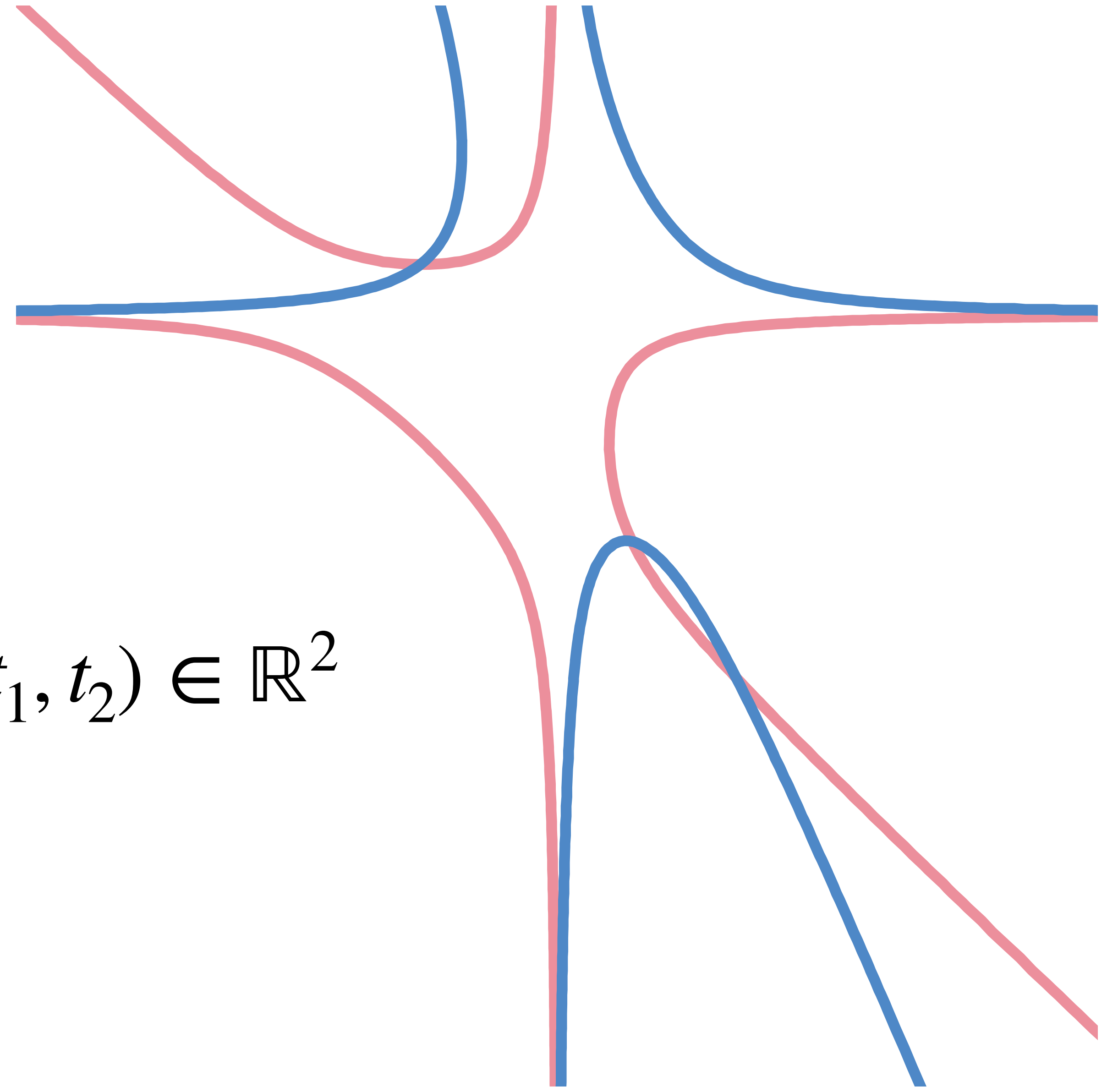

$(t_1, t_2) \in \mathbb{R}^2$

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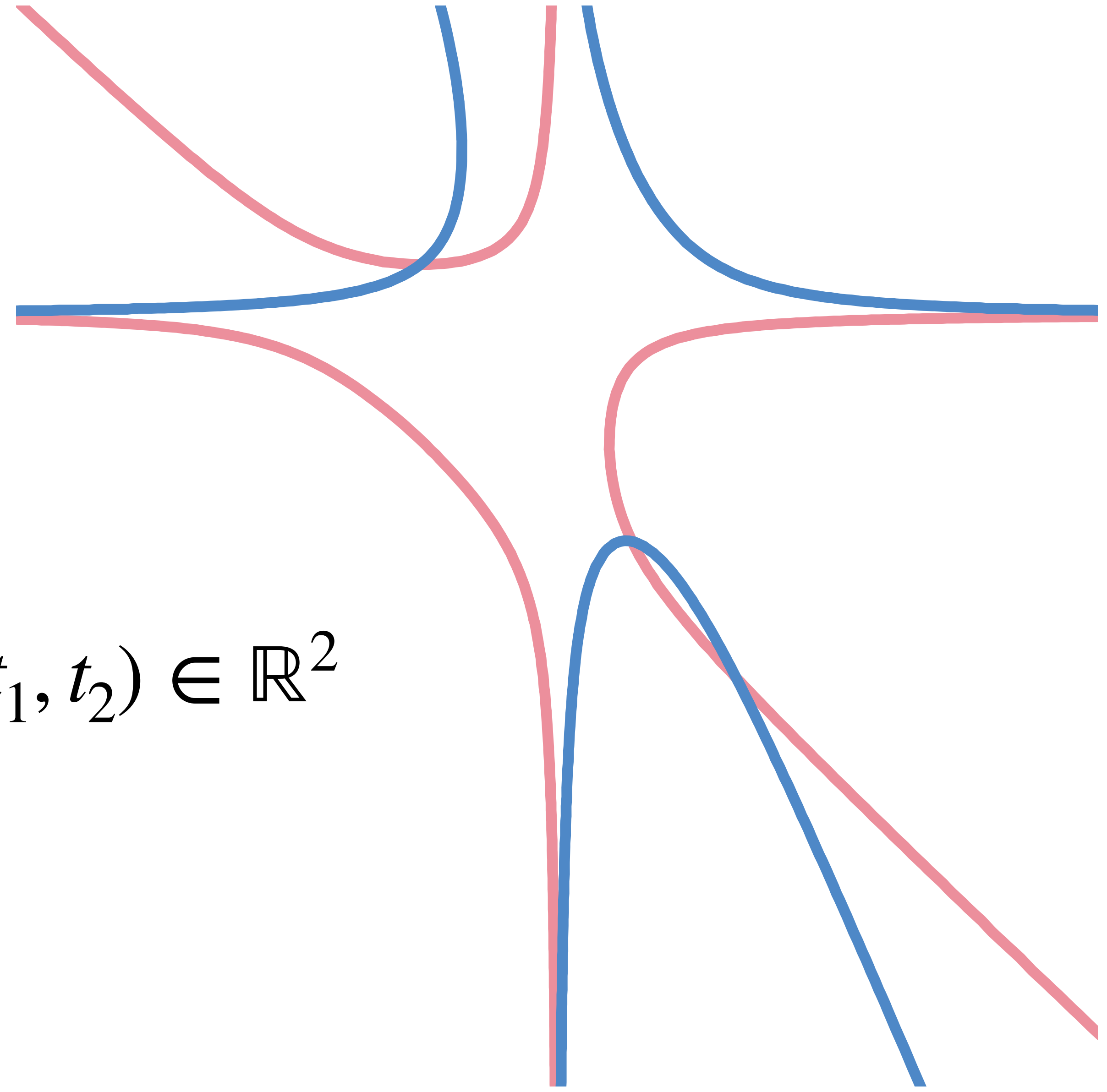

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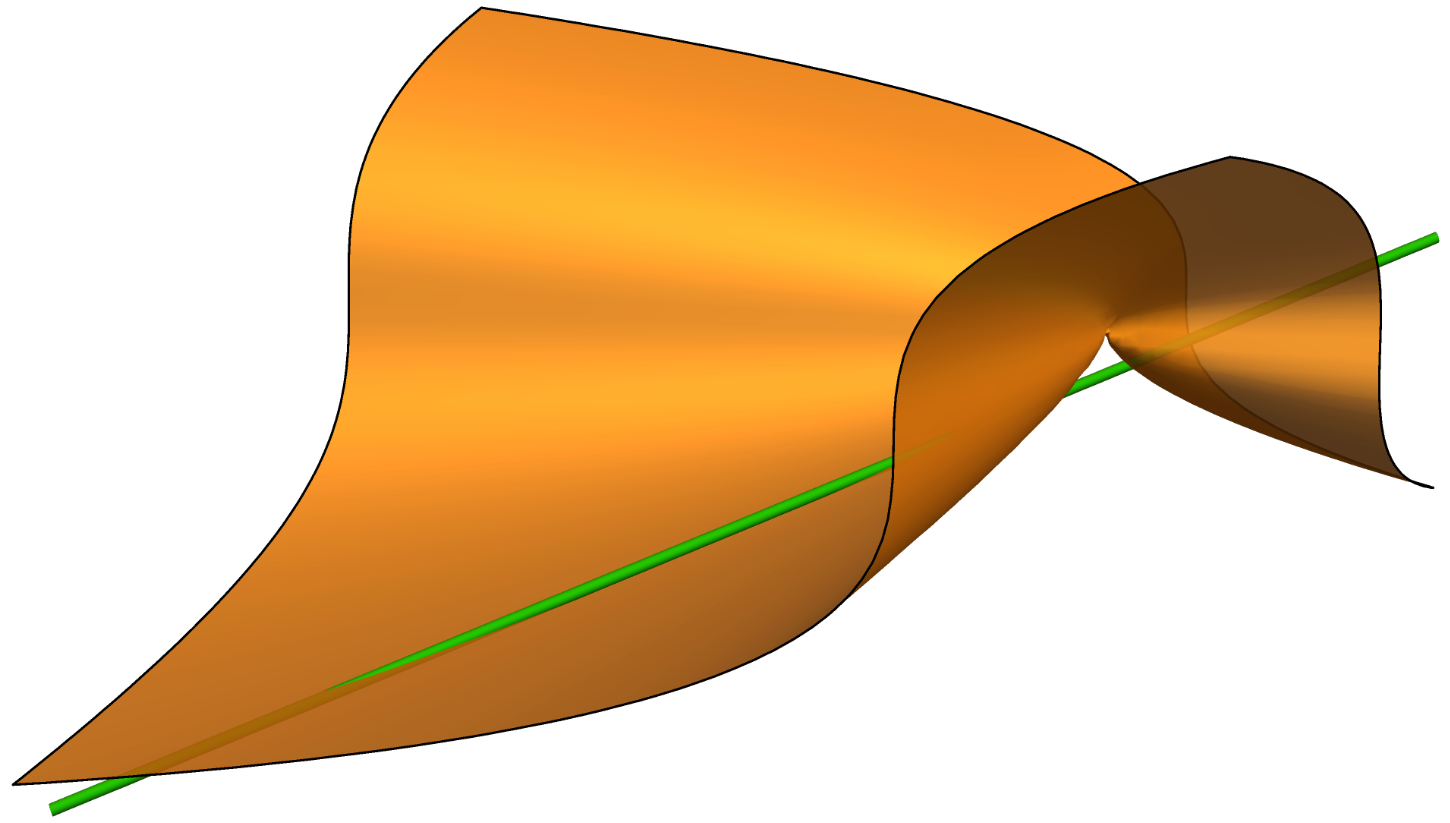

$$\begin{cases} 1 + 4x + 3y + 2z = 0, \\ 3 + 2x - 3y - 5z = 0 \end{cases}$$

$(x, y, z) \in Y_A = \{x^3 - yz = 0\}$

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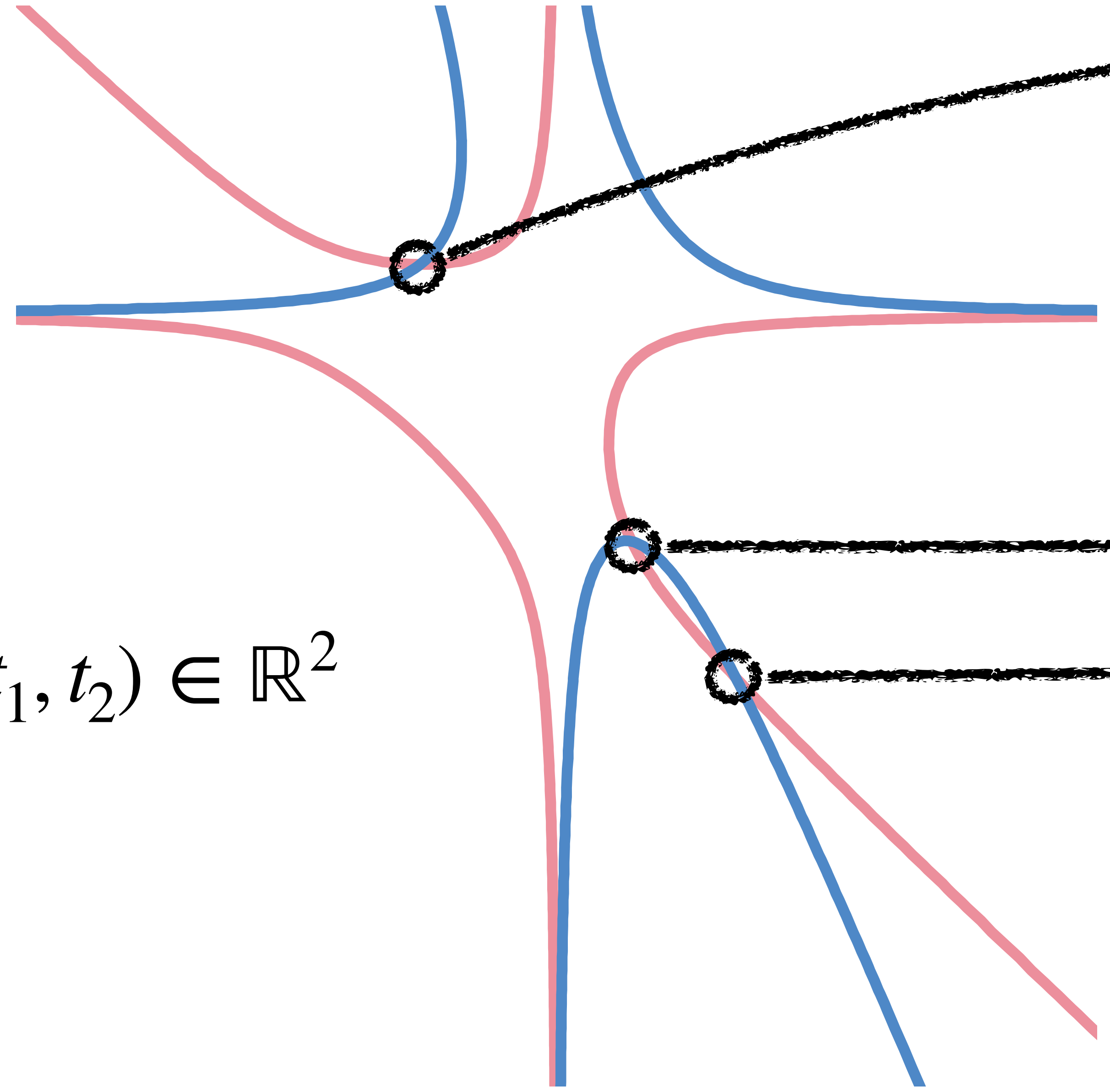

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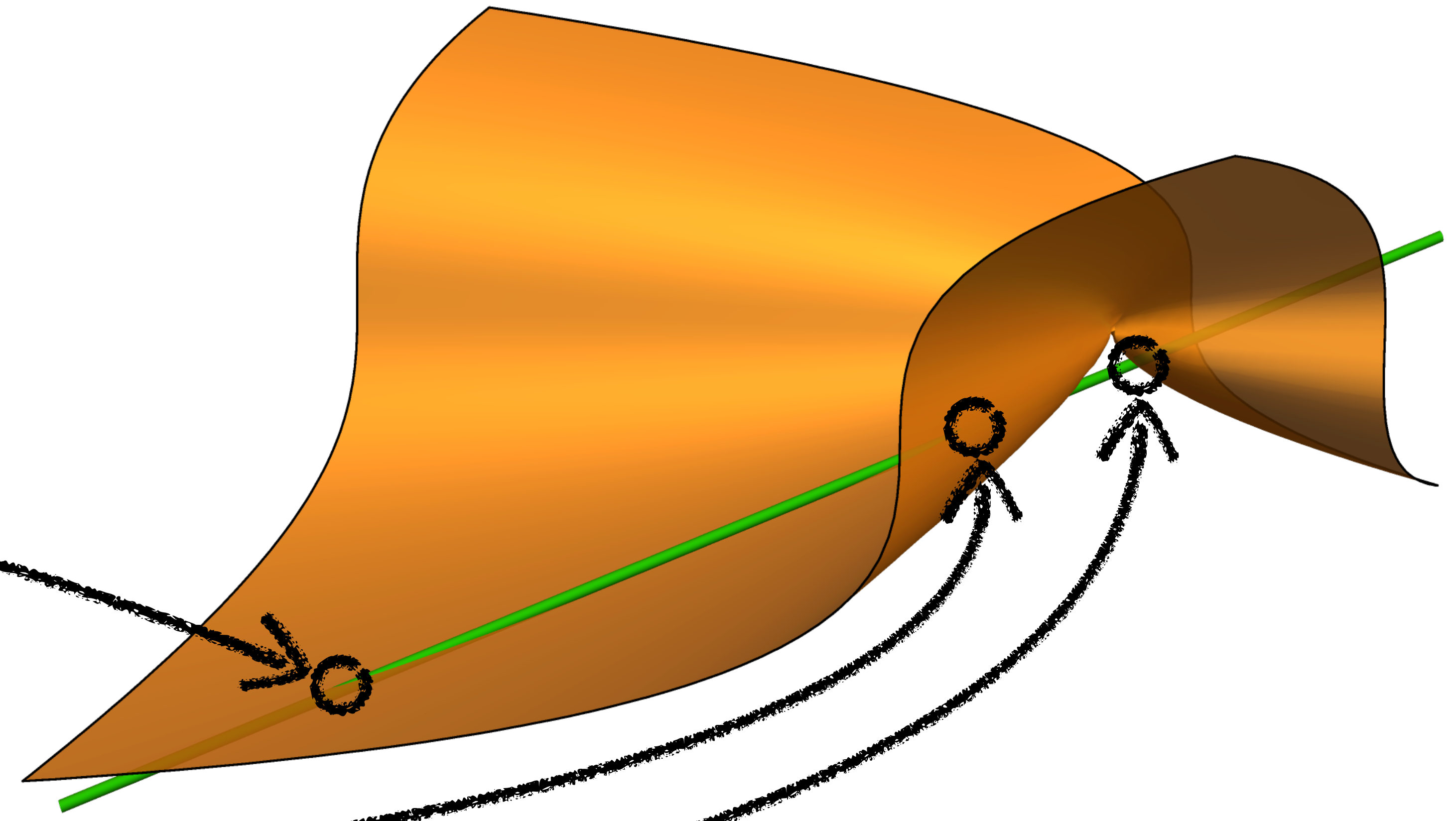

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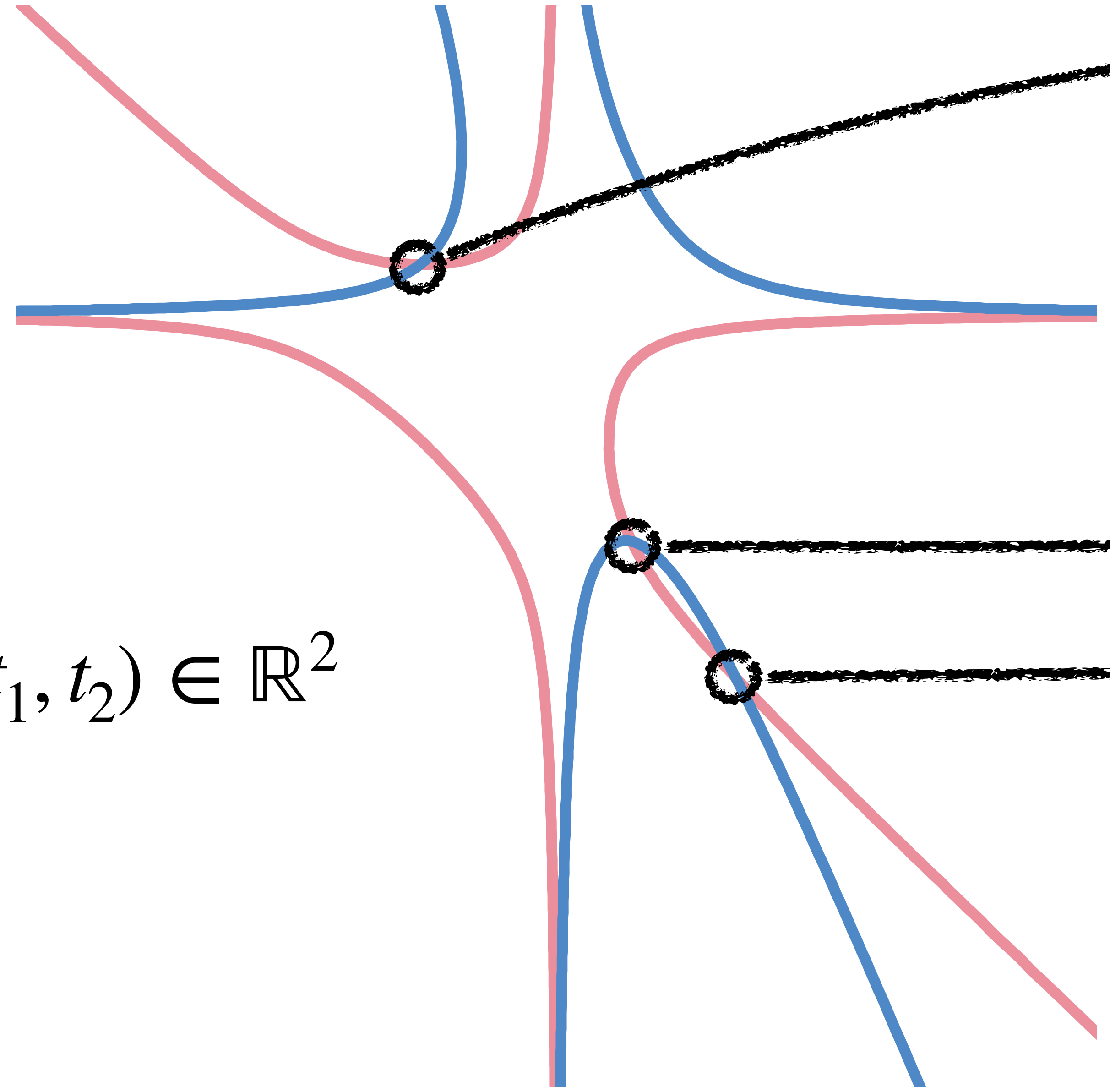

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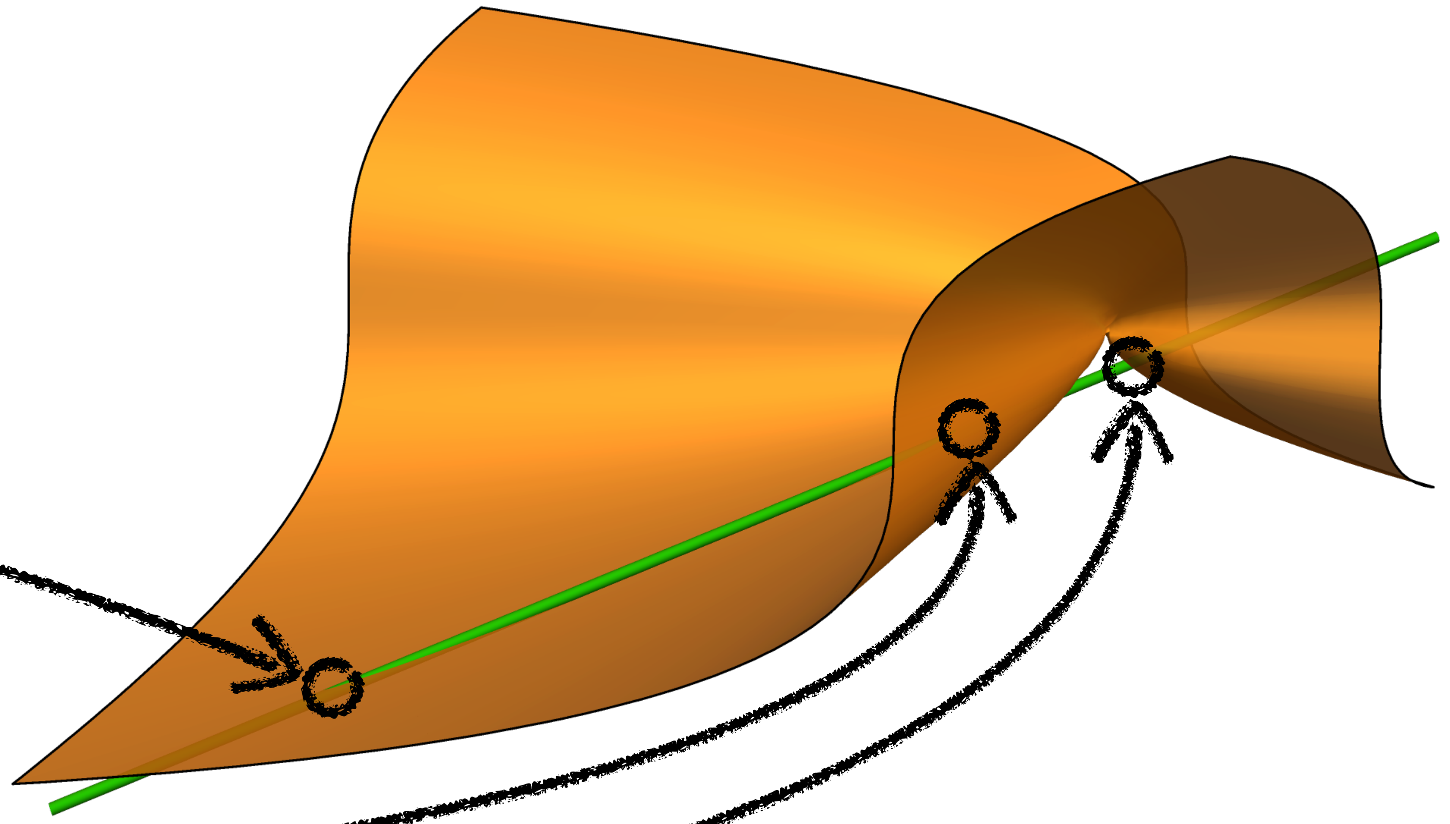
solutions $= \deg Y_A = \operatorname{vol} P$



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Why the monomial basis?

Solving polynomial systems in practice:

monomial basis: ill-conditioned

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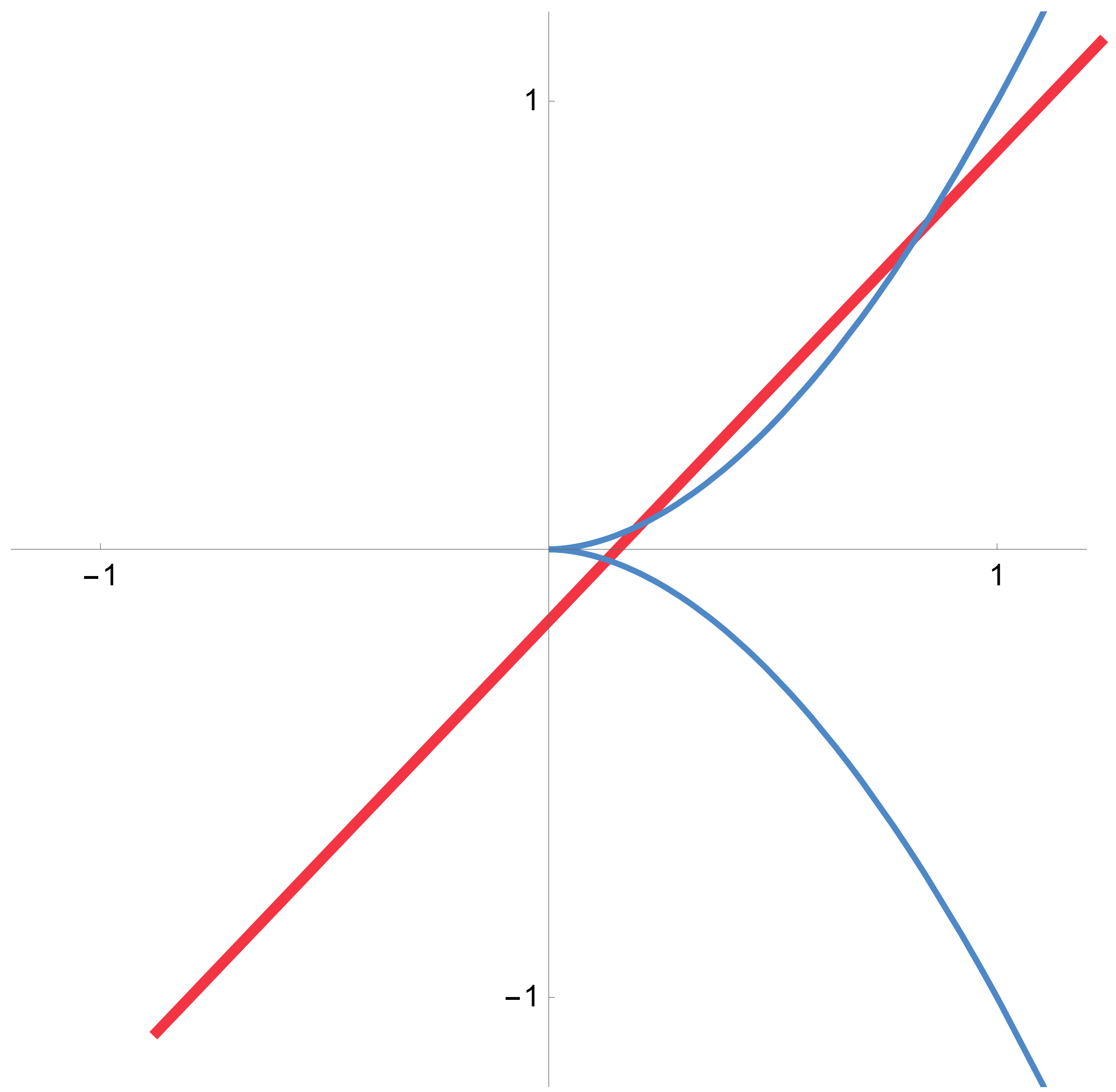
monomials : Toric varieties = Chebyshev polynomials : Chebyshev varieties

Transition

monomial basis

$$a + bt^6 + ct^{11} = 0$$

$a + bx + cy = 0$ for $(x, y) \in Y_{(6,11)}$



Transition

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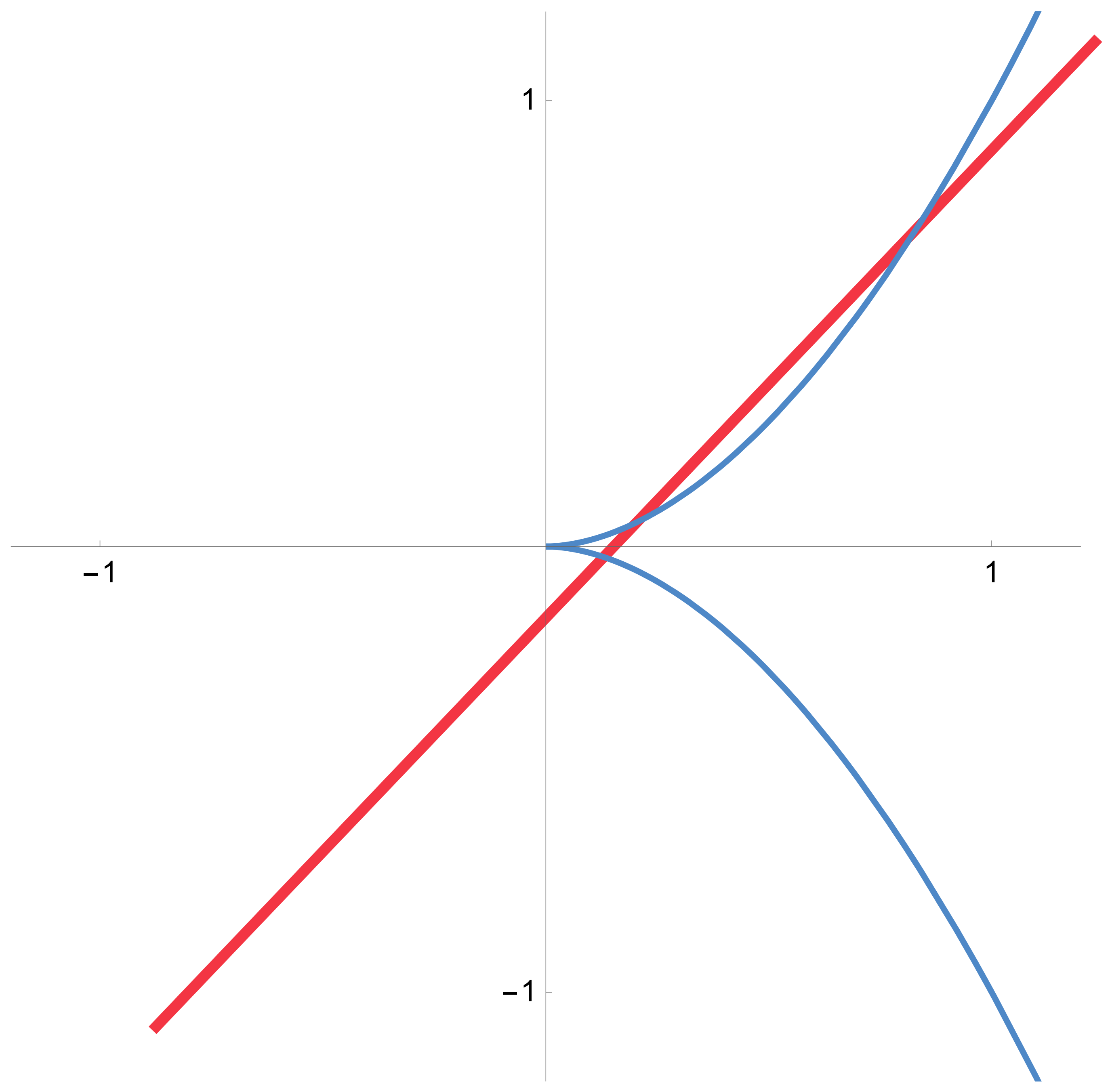
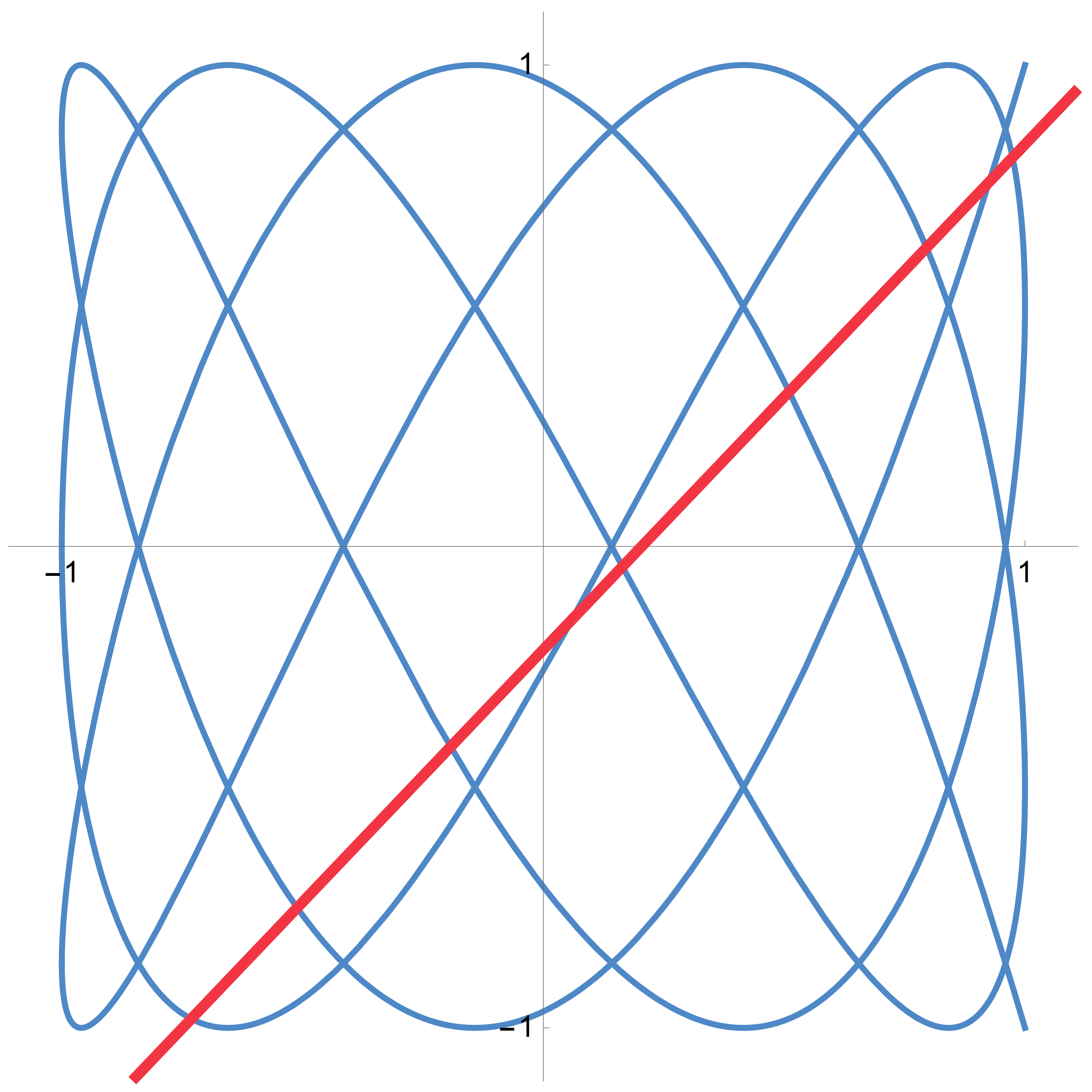
monomial basis	Chebyshev basis
$a + bt^6 + ct^{11} = 0$	$a + bT_6(t) + cT_{11}(t) = 0$
$a + bx + cy = 0$ for $(x, y) \in Y_{(6,11)}$	$a + bx + cy = 0$ for $(x, y) \in X_{(6,11)}$





Chebyshev polynomials

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$$T_0(x) = 1, \qquad T_1(x) = x,$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

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$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

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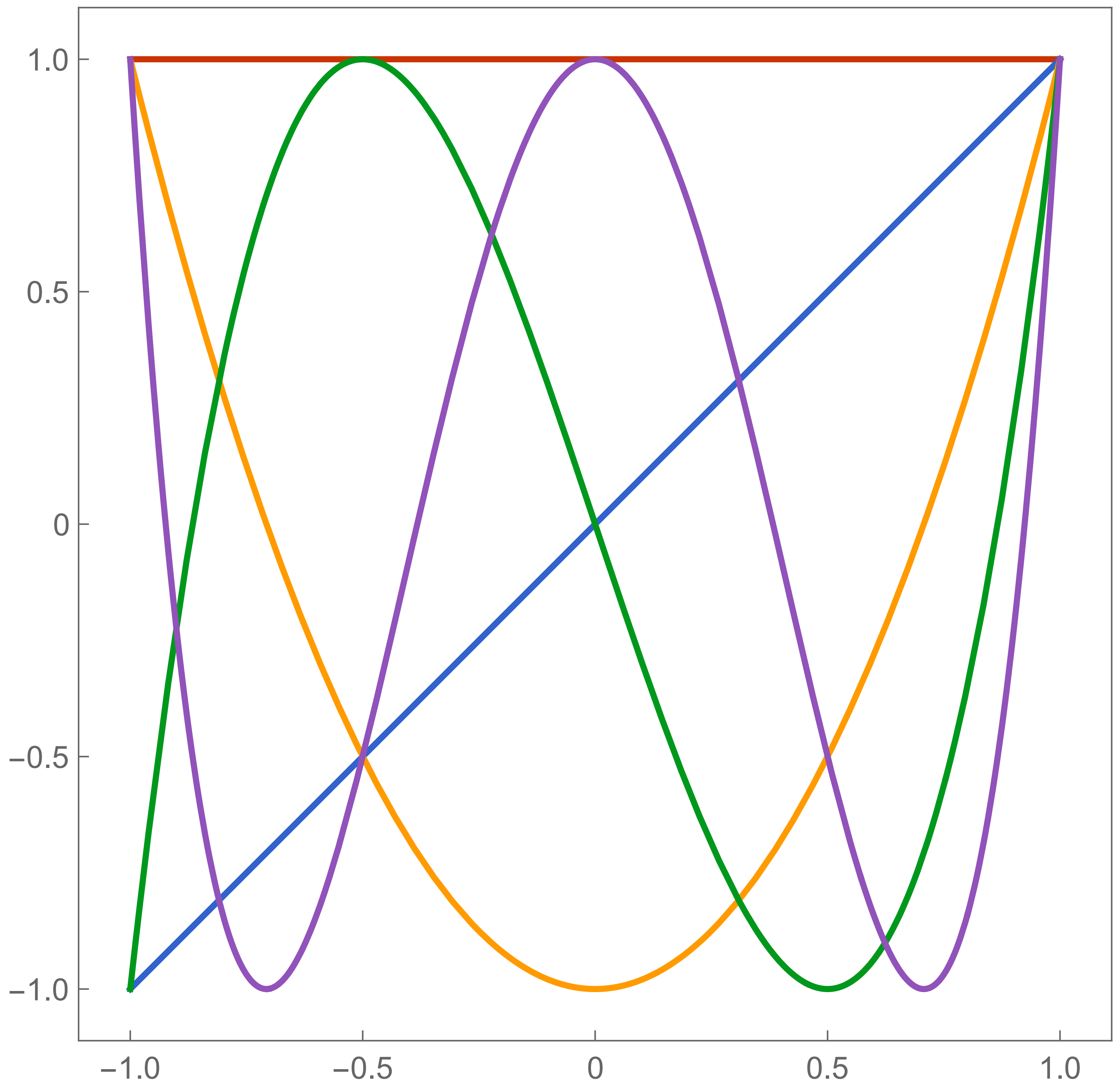
$T_4(x) = 8x^4 - 8x^2 + 1$

$\vdots$



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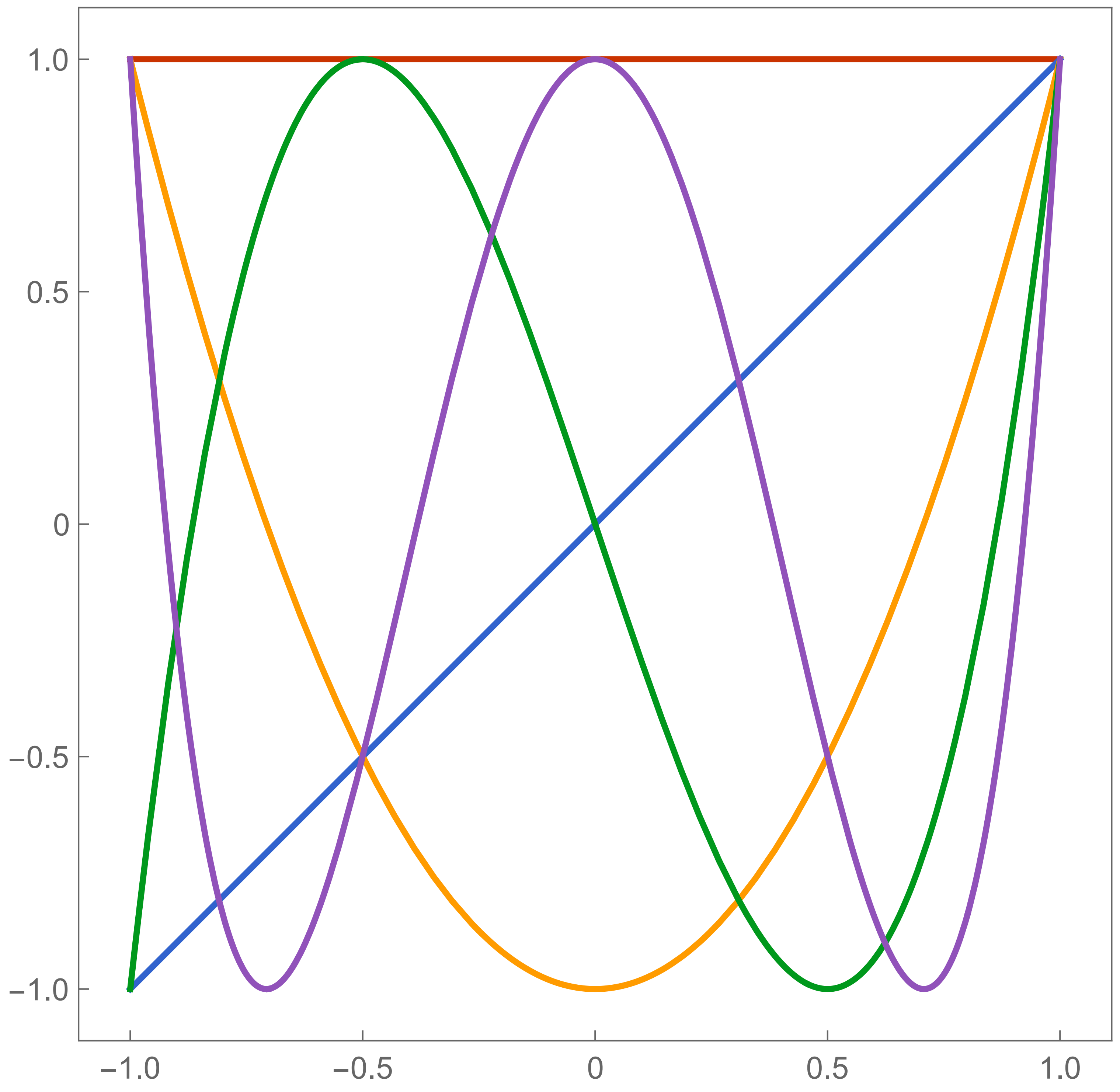

Monomial-like:

$$T_{n\cdot m}(x) = T_{m\cdot n}(x) = T_n(x)T_m(x)$$

Trigonometric-like:

$$T_n(x) = \cos(n \arccos x)$$

Orthogonal:

$$\int_{-1}^{1} T_n(x)T_m(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{\pi}{2}\delta_{n,m}$$

→ interlaced roots

Chebyshev curves

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Definition: Let $A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$ and consider the map $\phi_A : \mathbb{C} \ni t \mapsto \left(T_{\alpha_1}(t), \ldots, T_{\alpha_s}(t)\right) \in \mathbb{C}^s$.
The Chebyshev curve X_A associated to A is the Zariski closure of the image of ϕ_A.

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Theorem (Freudenburg x2): If $A = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ are coprime, then $X_A = \{T_{\alpha_2}(x) - T_{\alpha_1}(y) = 0\}$ and it is irreducible. All its singularities are nodes.

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Theorem (Bel-Afia, M., Telen):
If $A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$ and at least three entries are pairwise coprime, then $X_A \subset \mathbb{C}^s$ is a smooth irreducible curve.

→ Equations ✓

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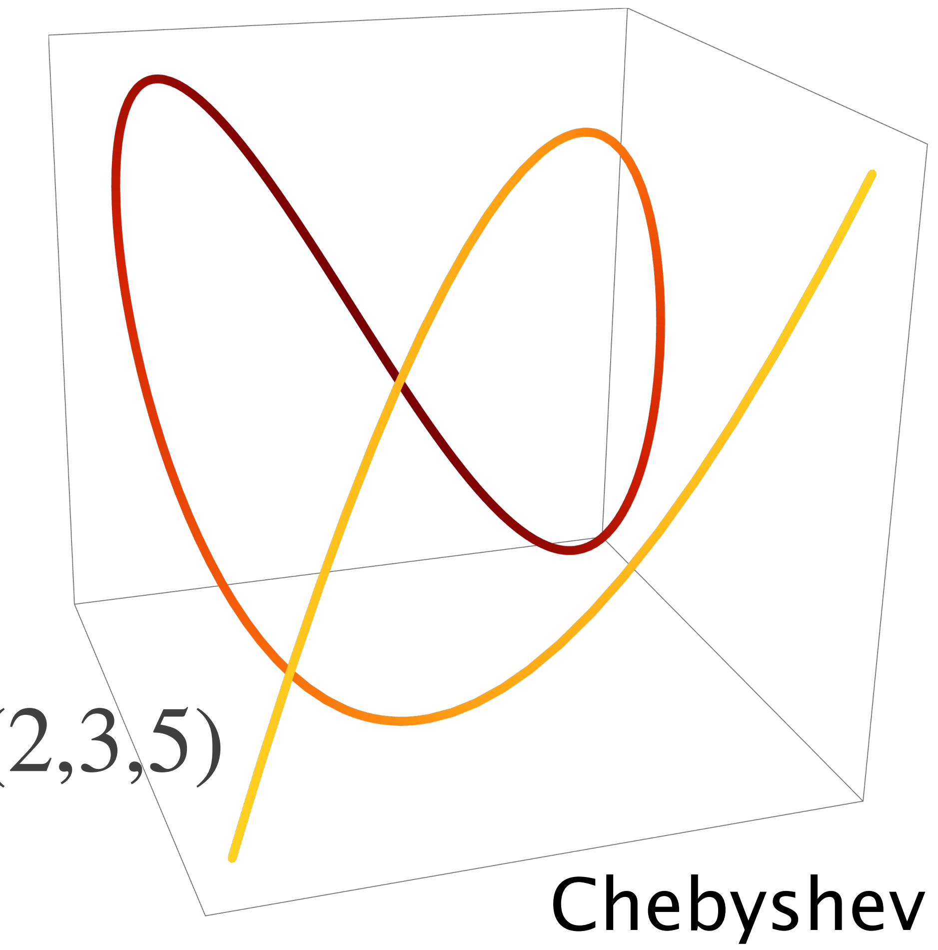
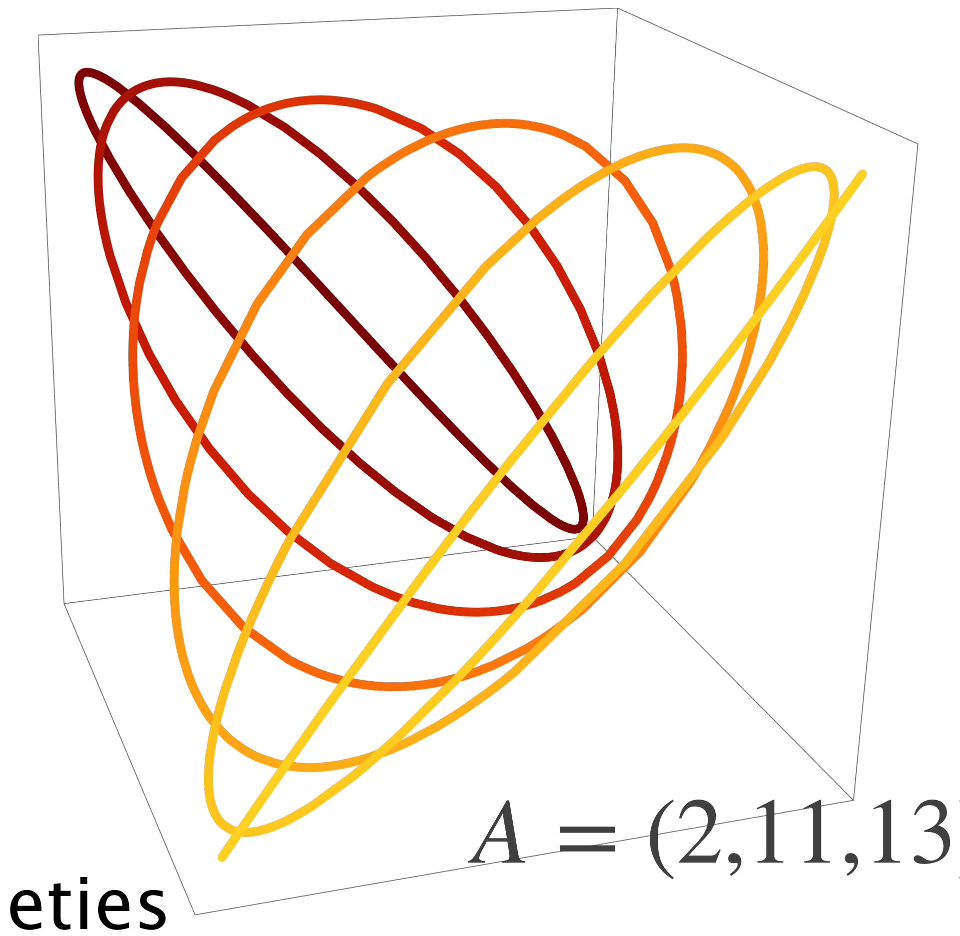
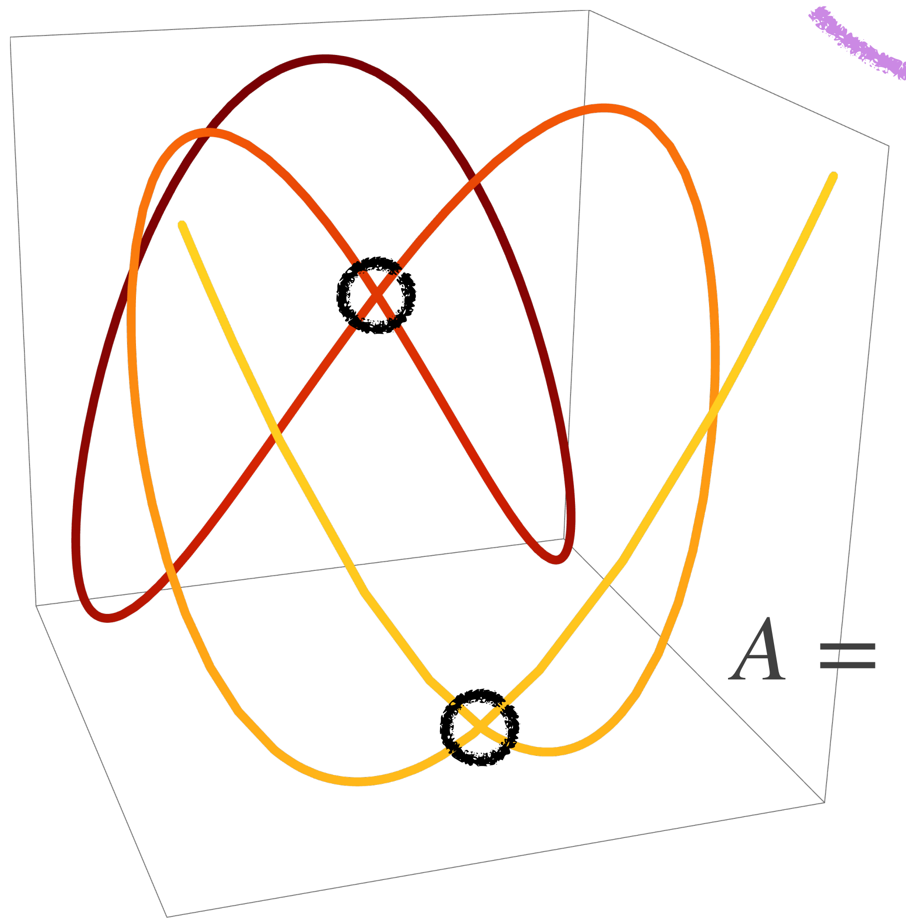


$A = (2,3,5)$



$A = (2,11,13)$



$A = (2,5,8)$

Equations ✓

ℝeal points in the plane

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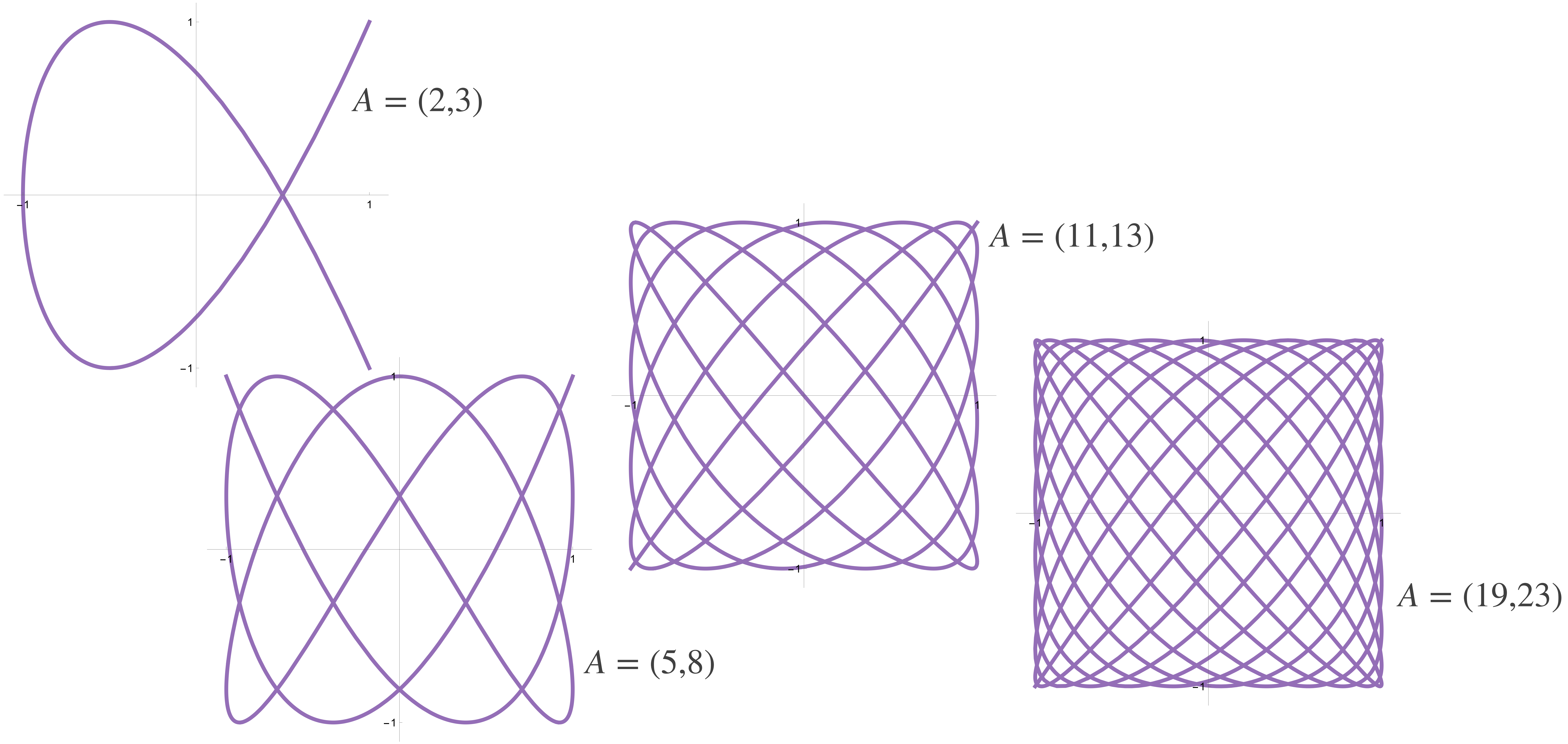
Q: What about real solutions?



$\mathbb{R}$eal points in the plane

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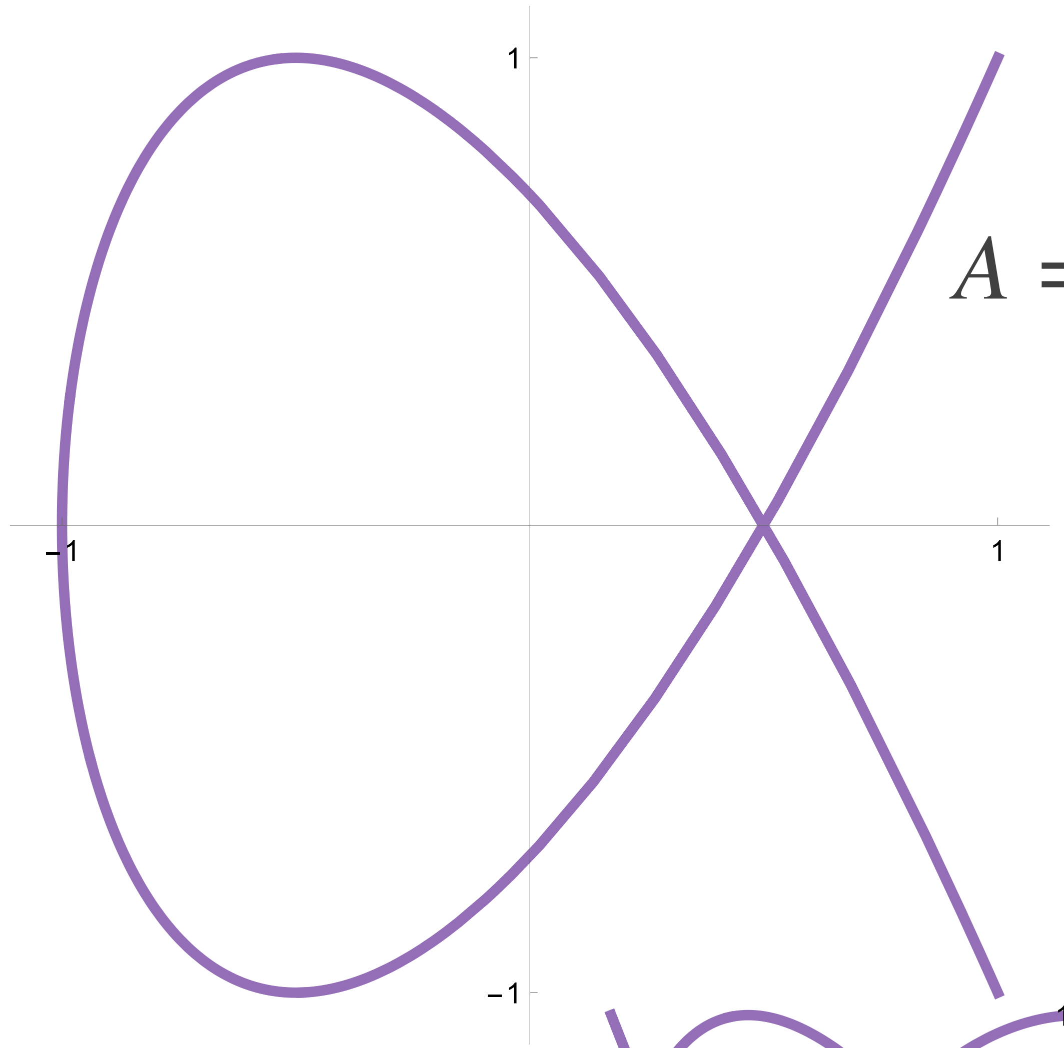
Q: What about real solutions?

Theorem (Bel-Afia, M., Telen):
Chebyshev curves $X_{(\alpha,\alpha+1)}$ are hyperbolic with respect to the origin. Namely,

$$aT_\alpha(t) + bT_{\alpha+1}(t) = 0$$

has only real solutions.



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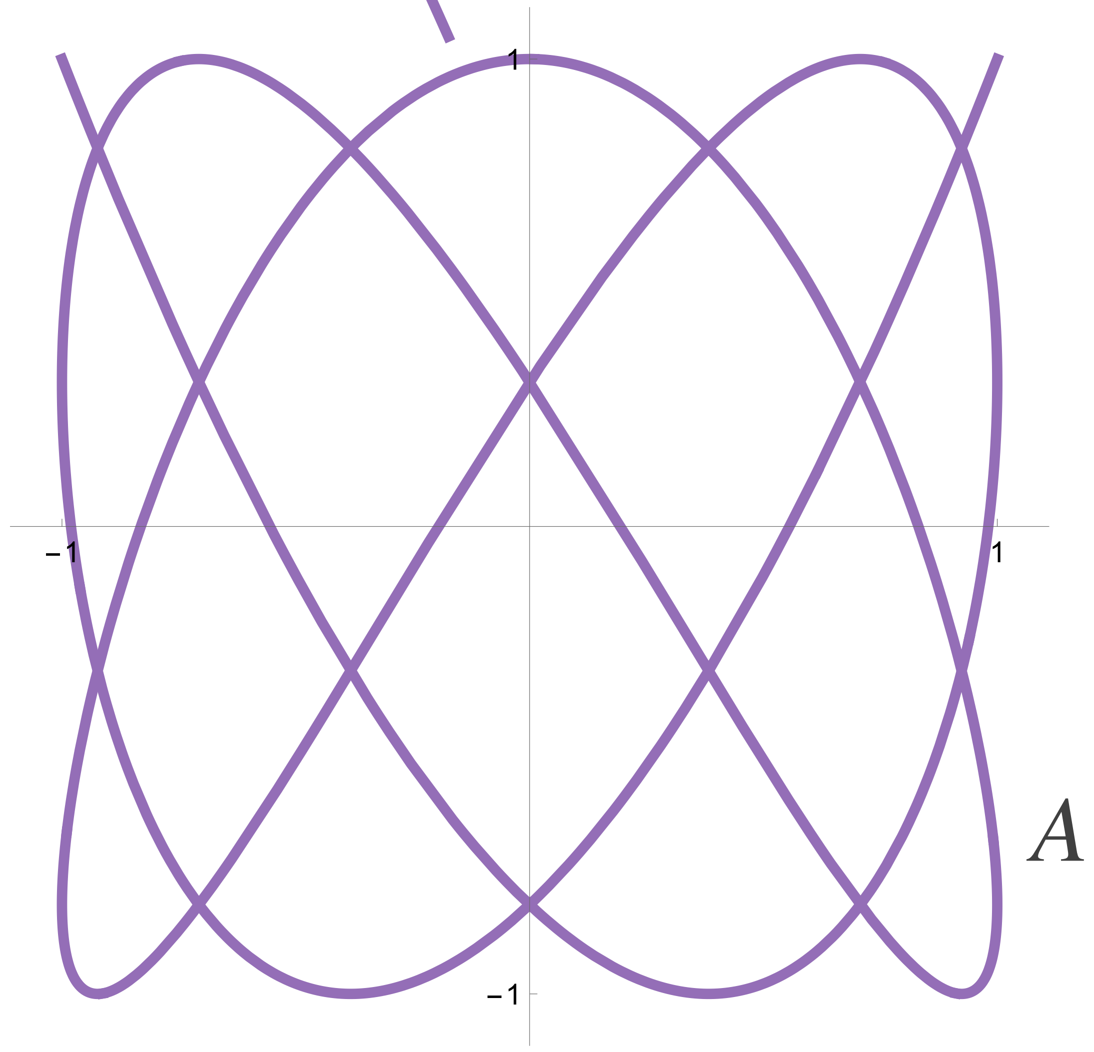
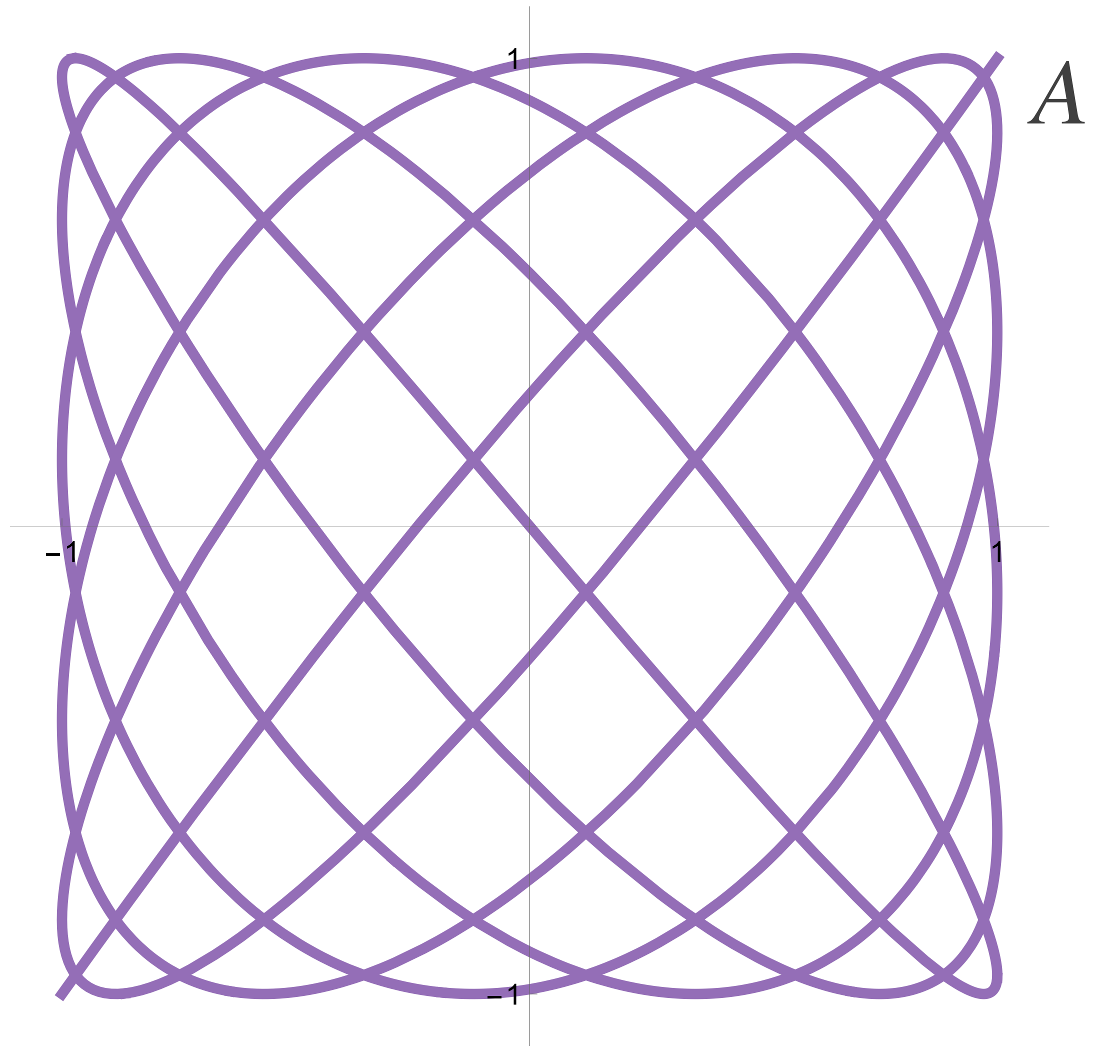


$A = (5,8)$



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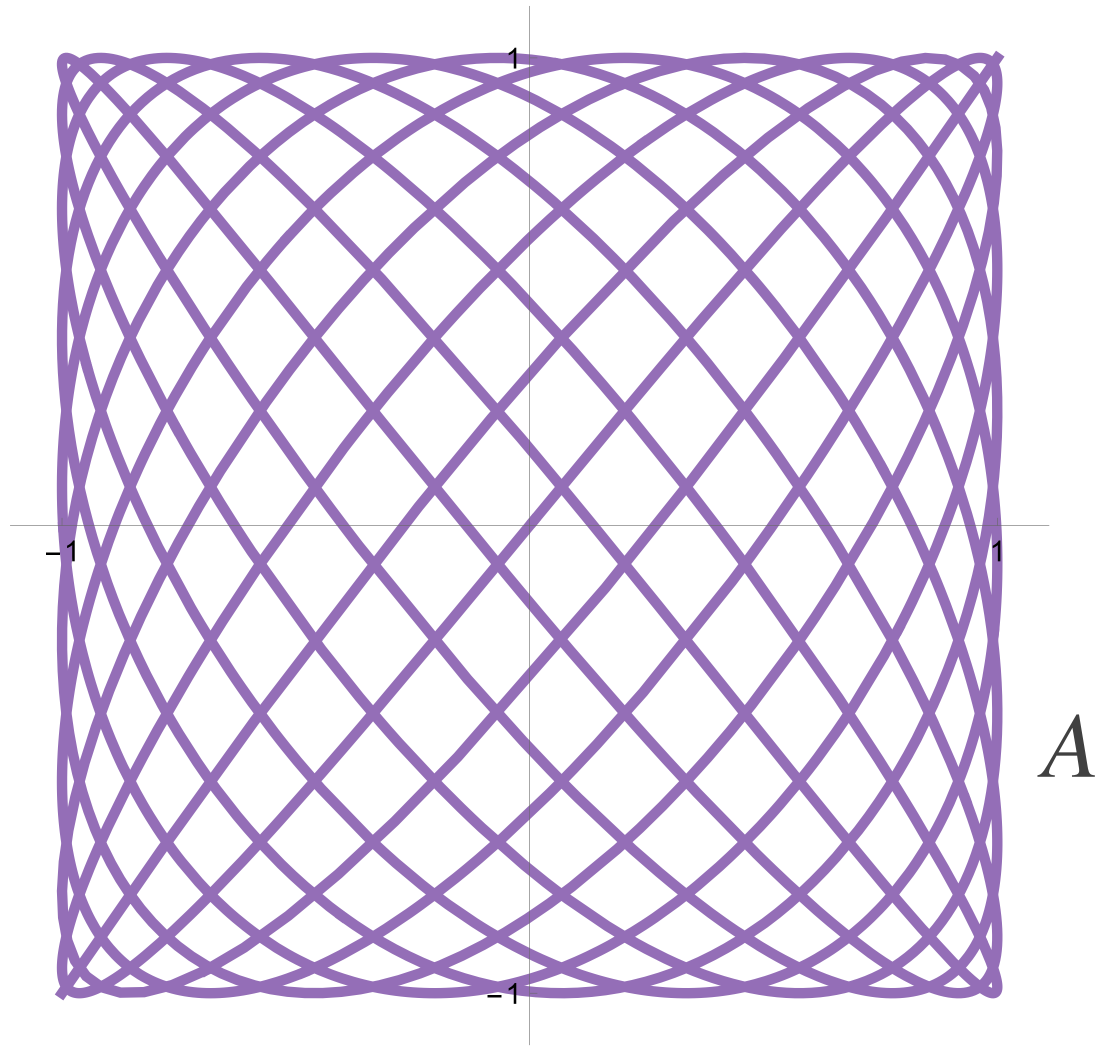

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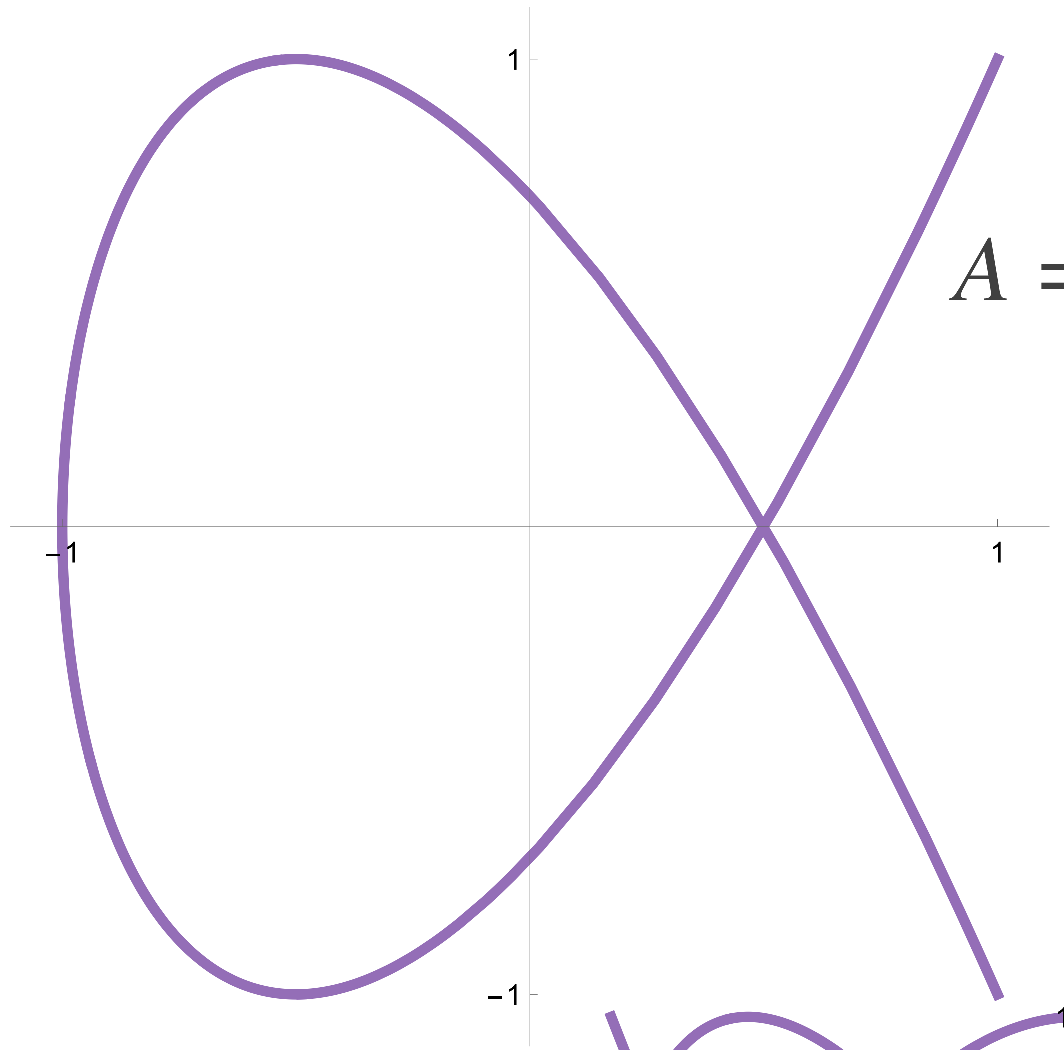
Lower bound of α_1 for $X_{(\alpha_1,\alpha_2)}$ assuming $\alpha_1 \leq \alpha_2$



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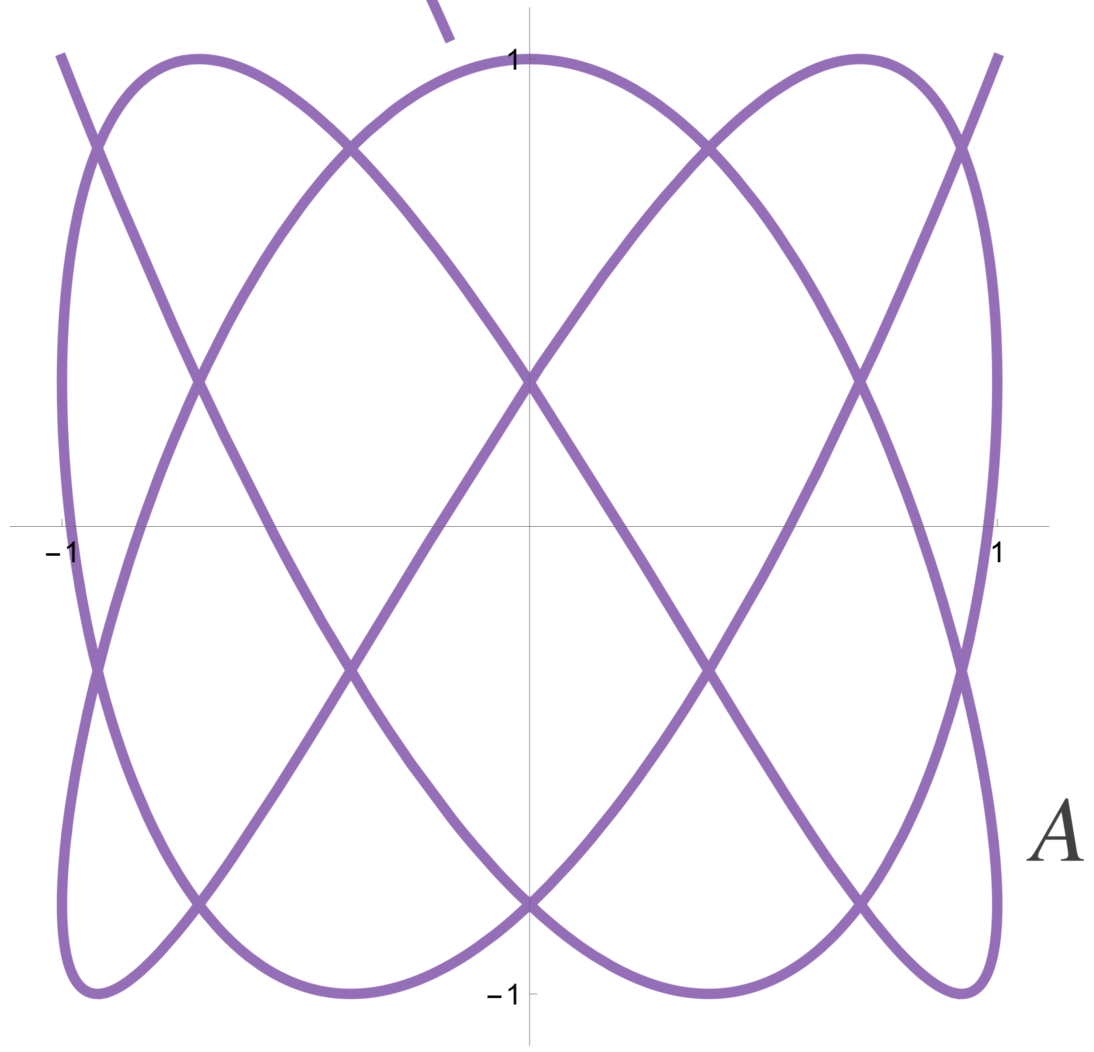

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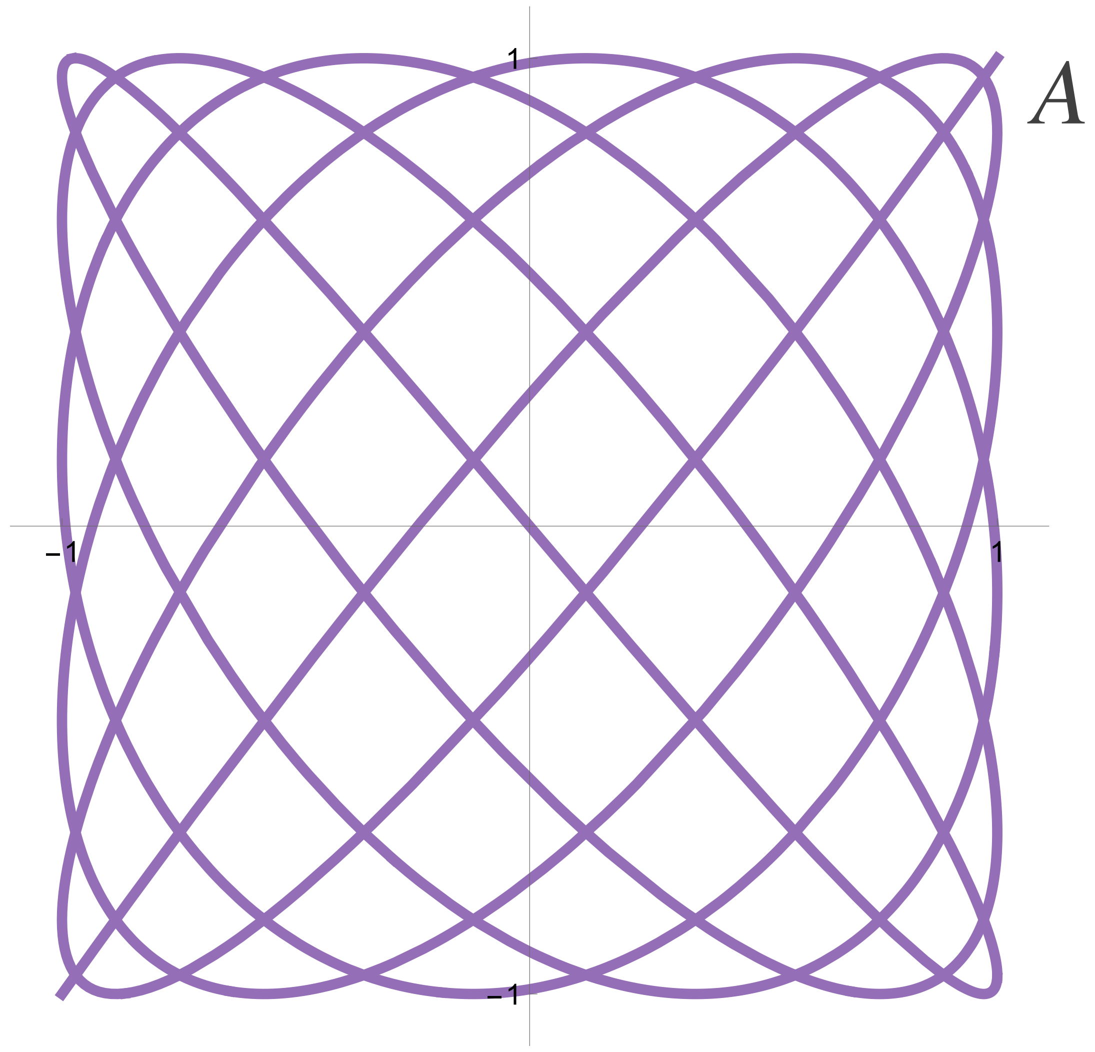

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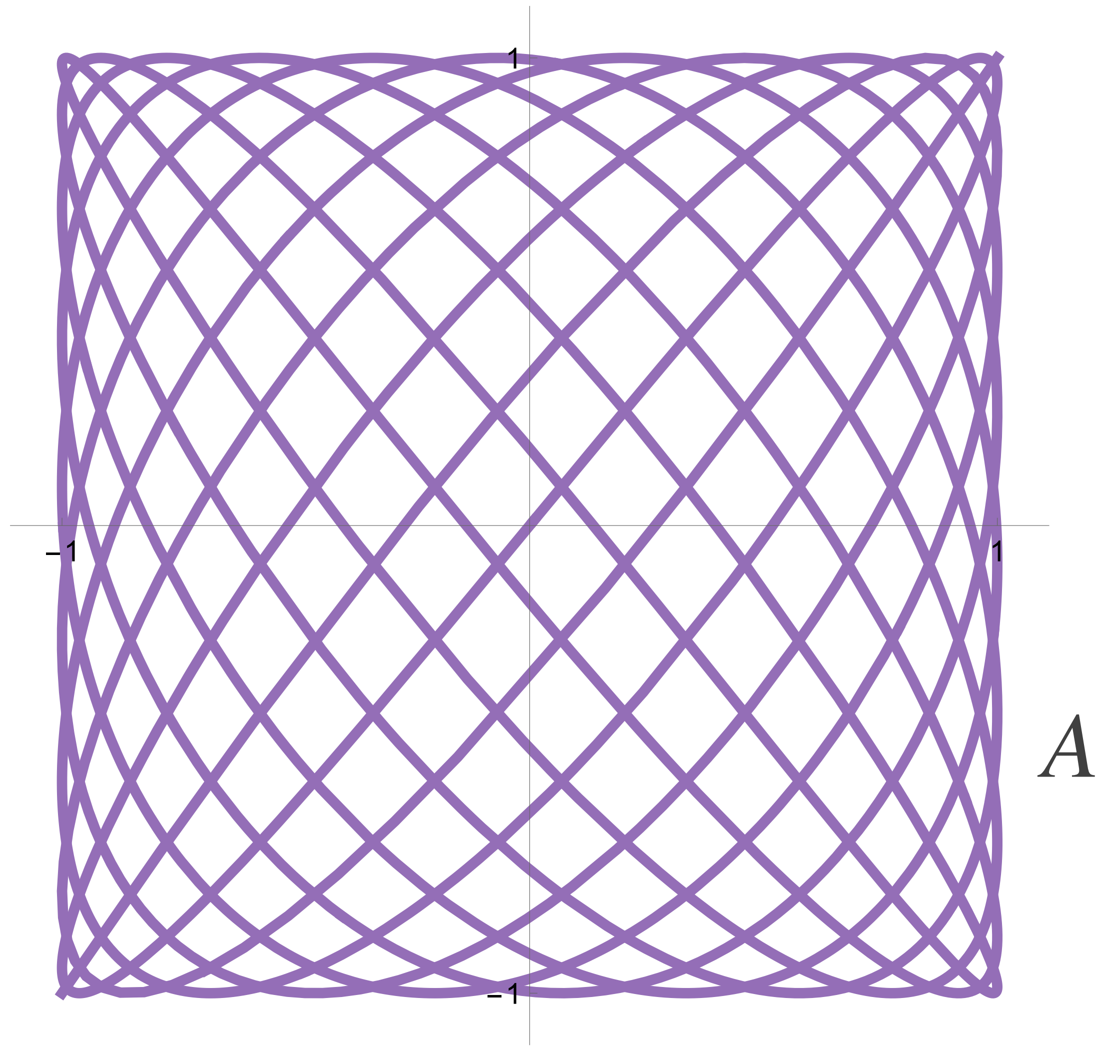
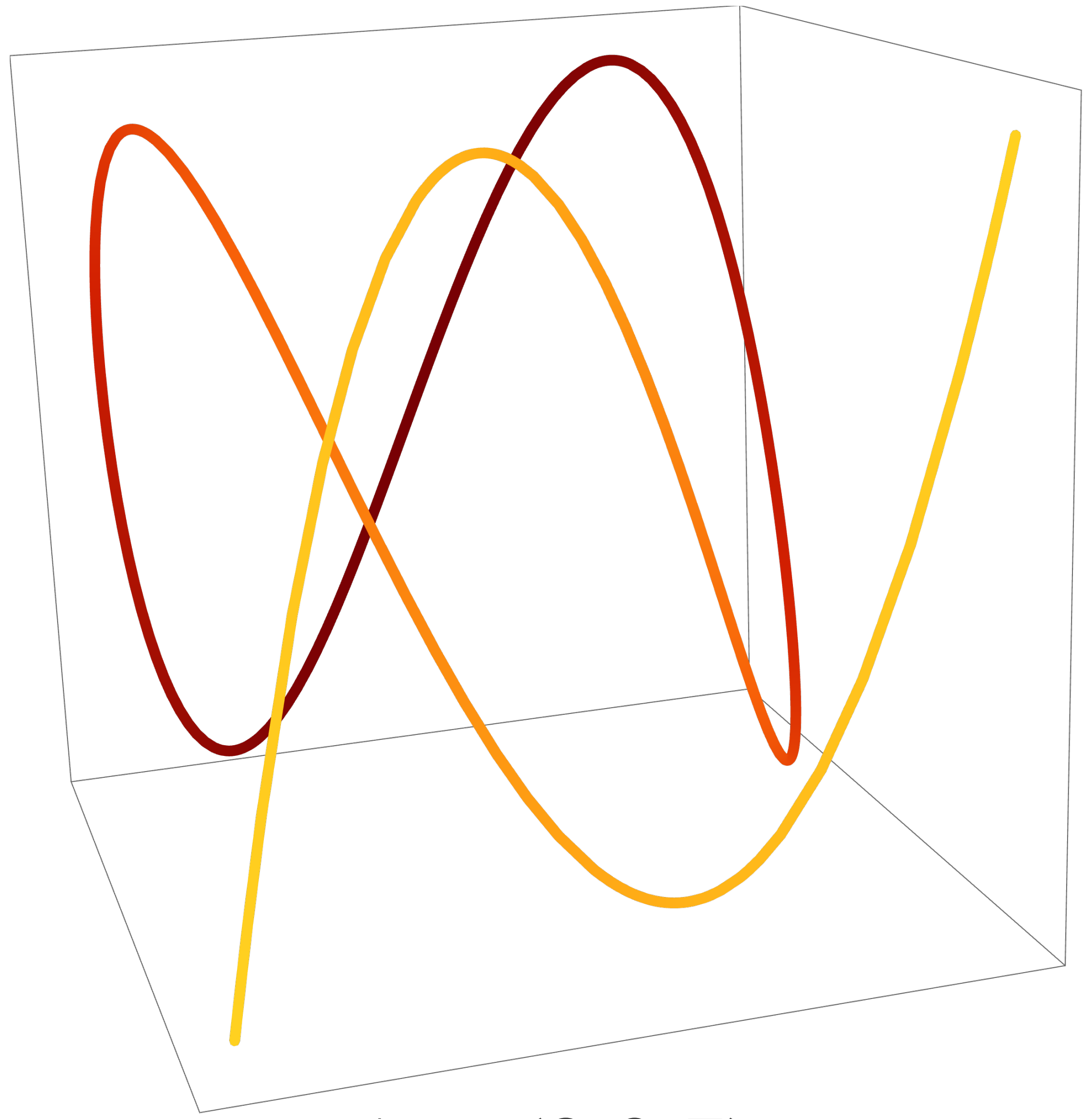


$A = (19,23)$

Experiment: ℝeal points

Let $v \in S^{s-1}$. How many real points does $X_A \cap v^\perp$ have?



$A = (2,3,7)$

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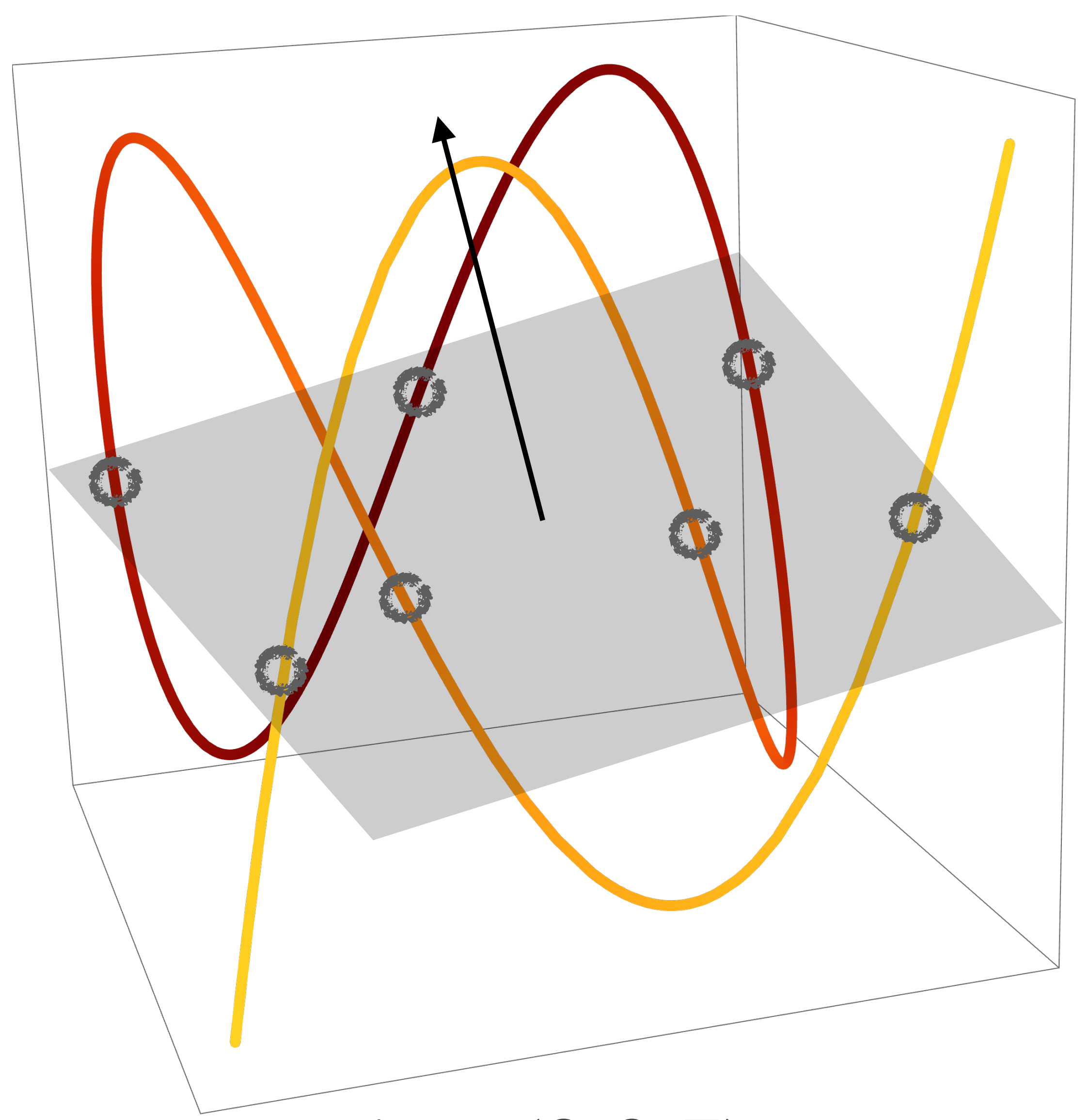

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Experiment: $\mathbb{R}$eal points

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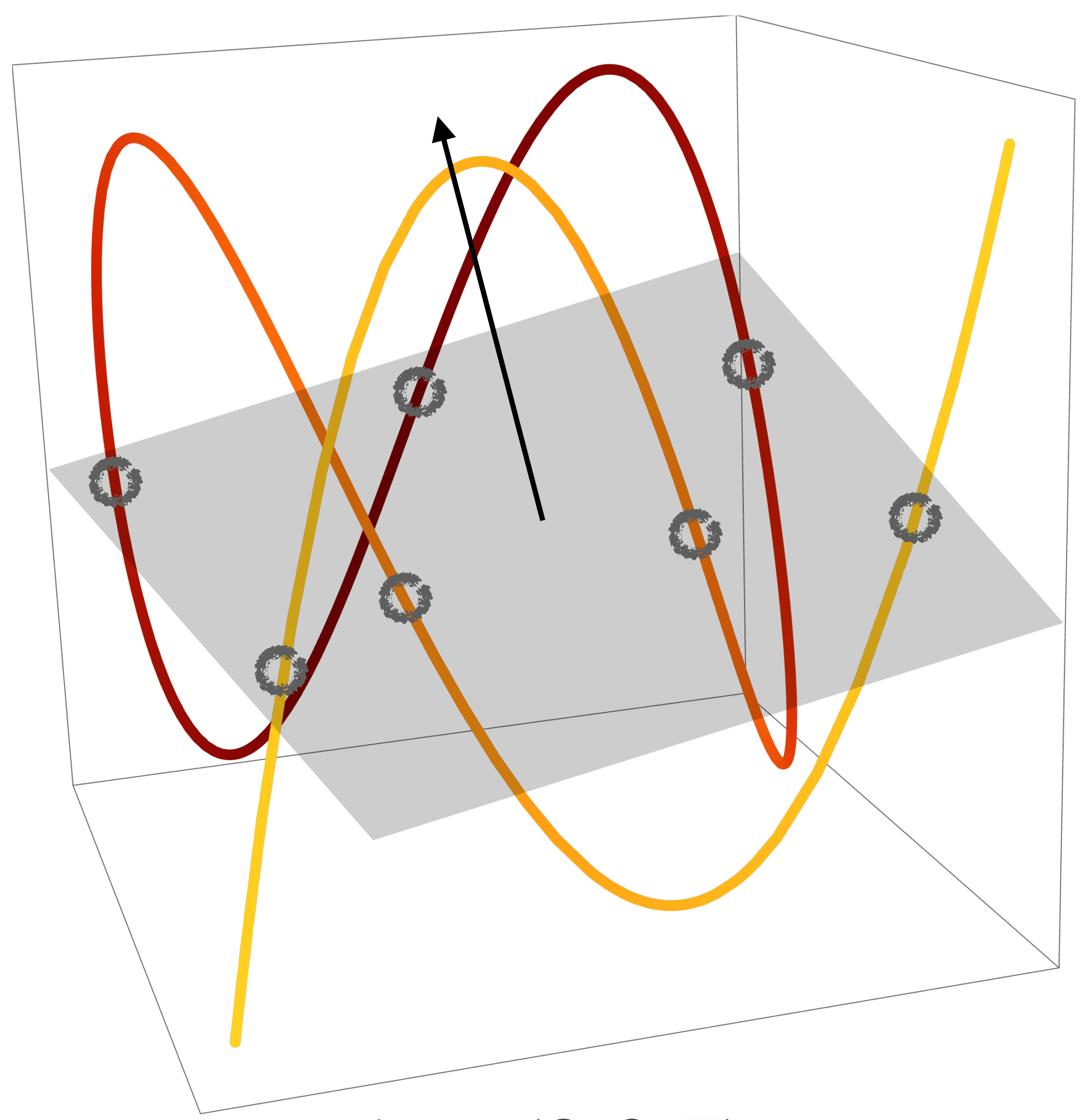

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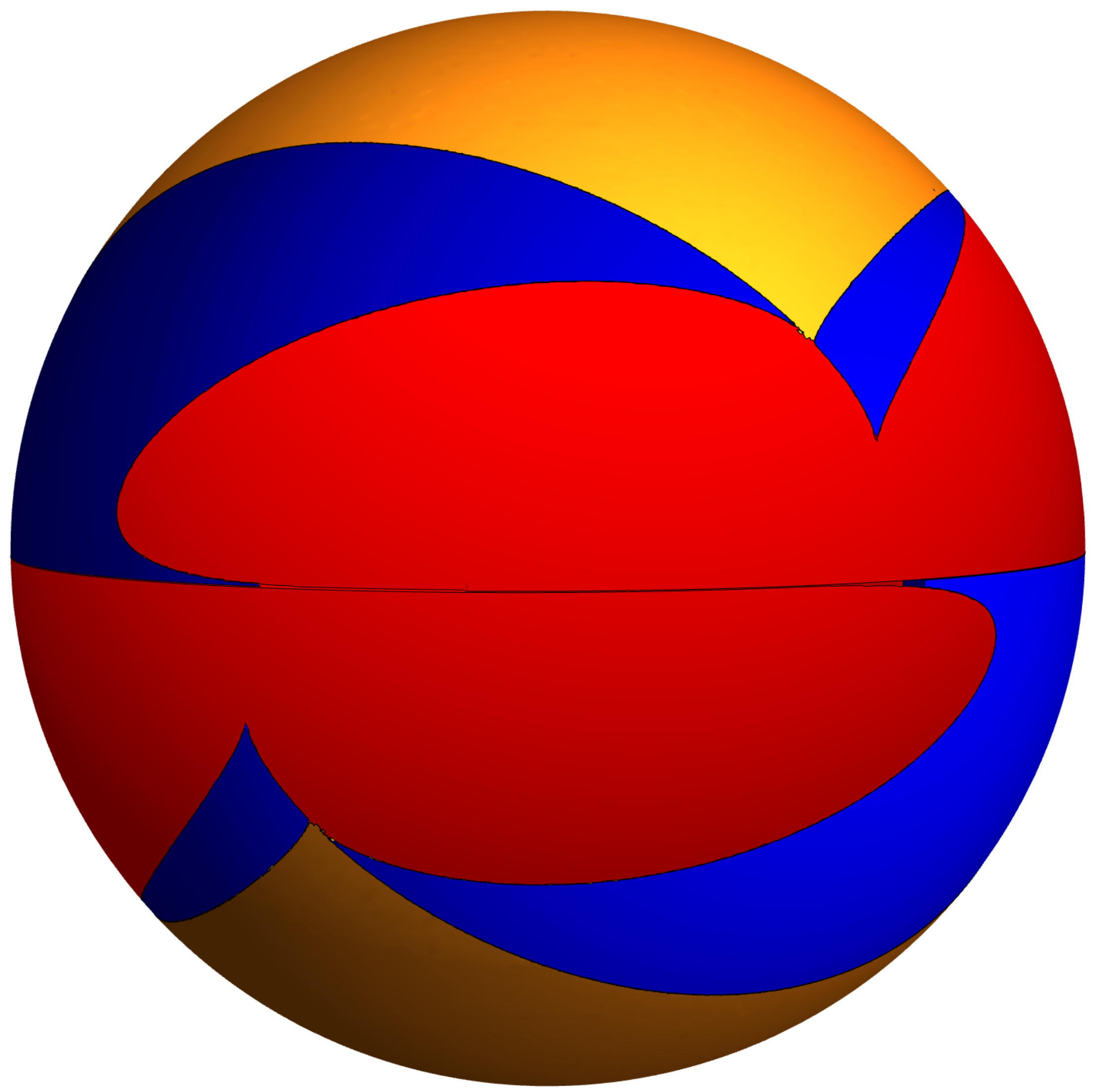
The real points of $X_A \cap v^{\perp}$ are at least 2 and at most 7.



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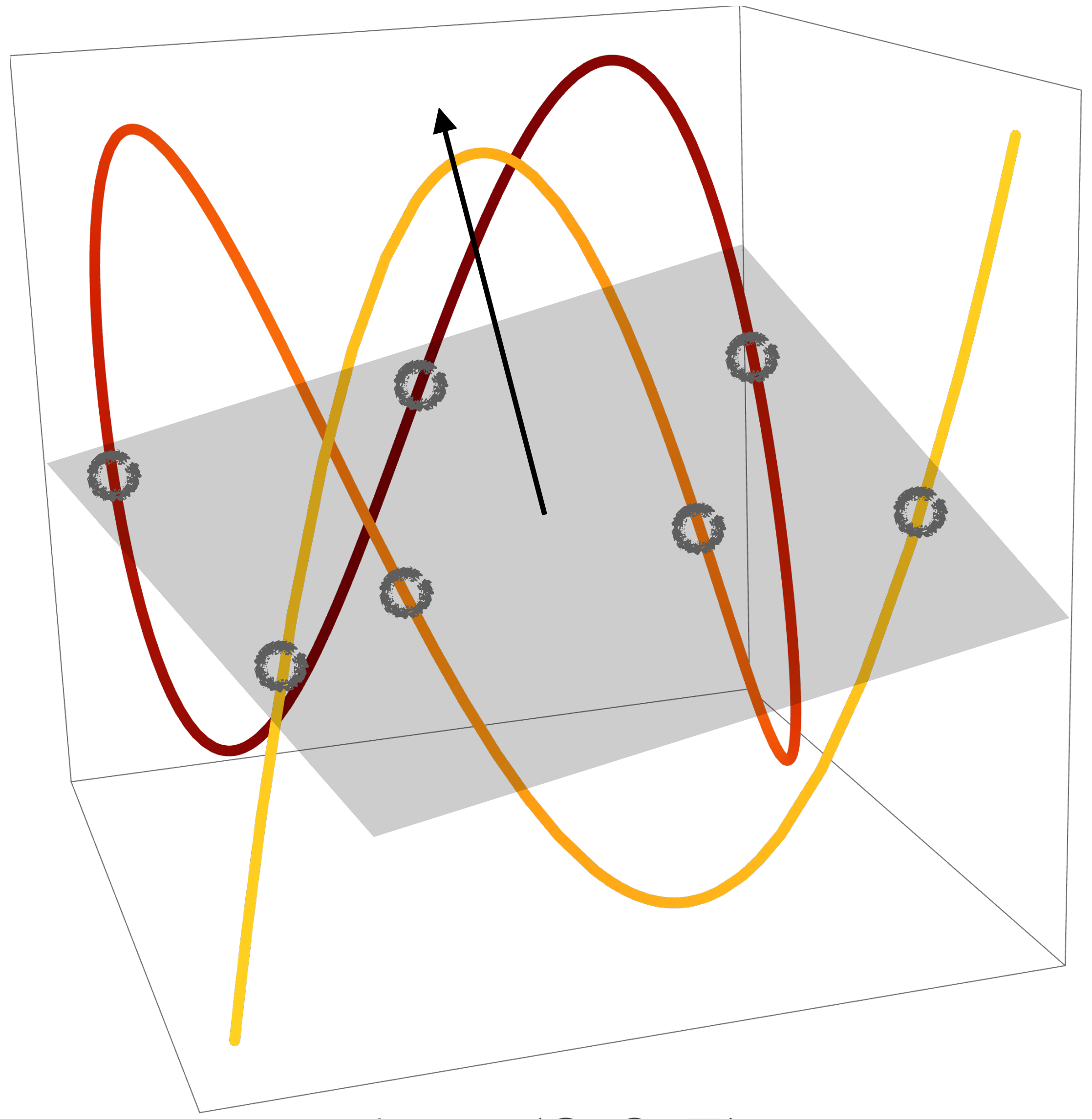

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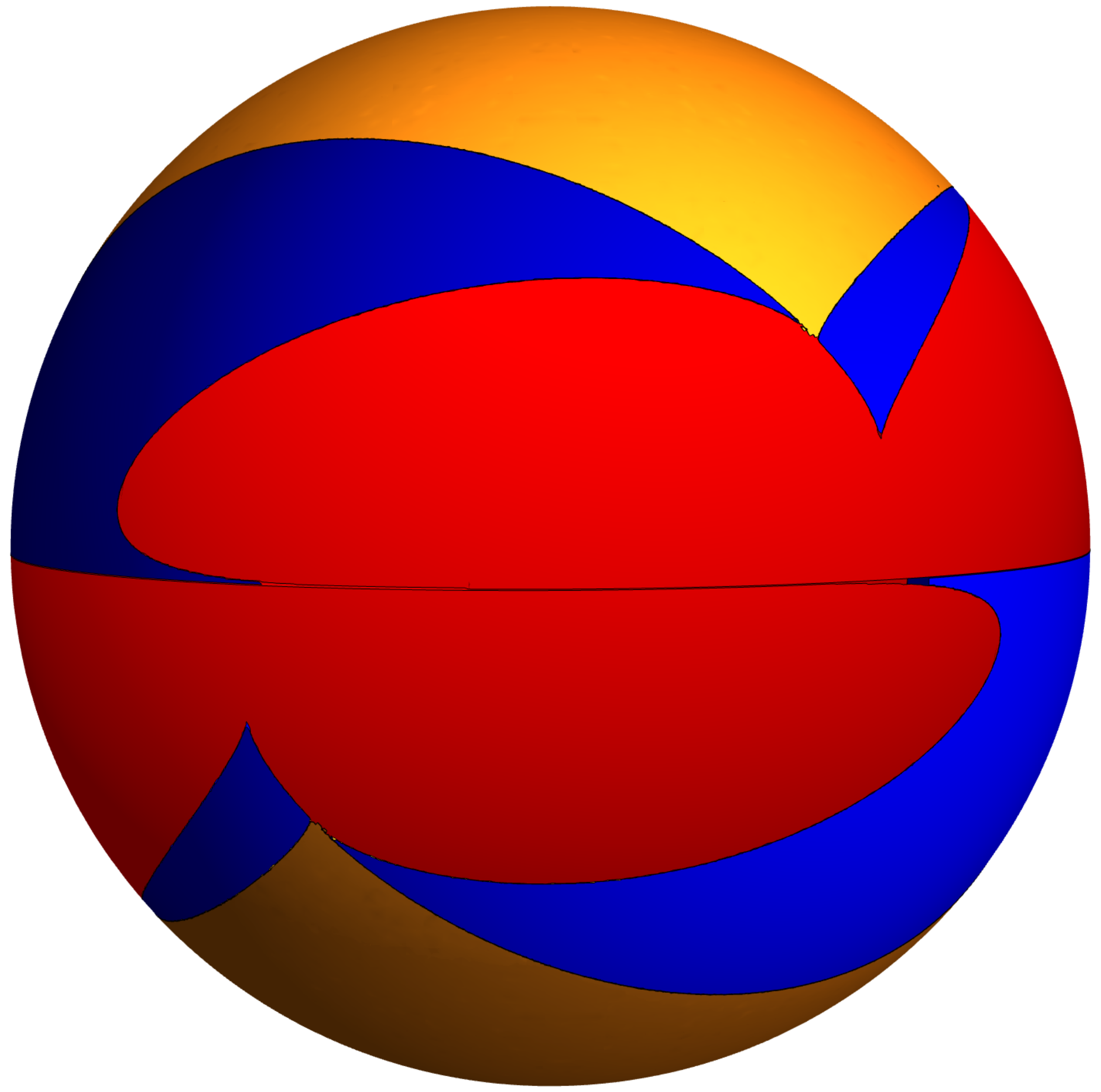

Conjecture: Let $A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$ be such that 3 of its entries are pairwise coprime. Any hyperplane through the origin intersects X_A in at least $\min_{j \in [s]} \alpha_j$ real points.

Chebyshev varieties

$A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^{n \times s}$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

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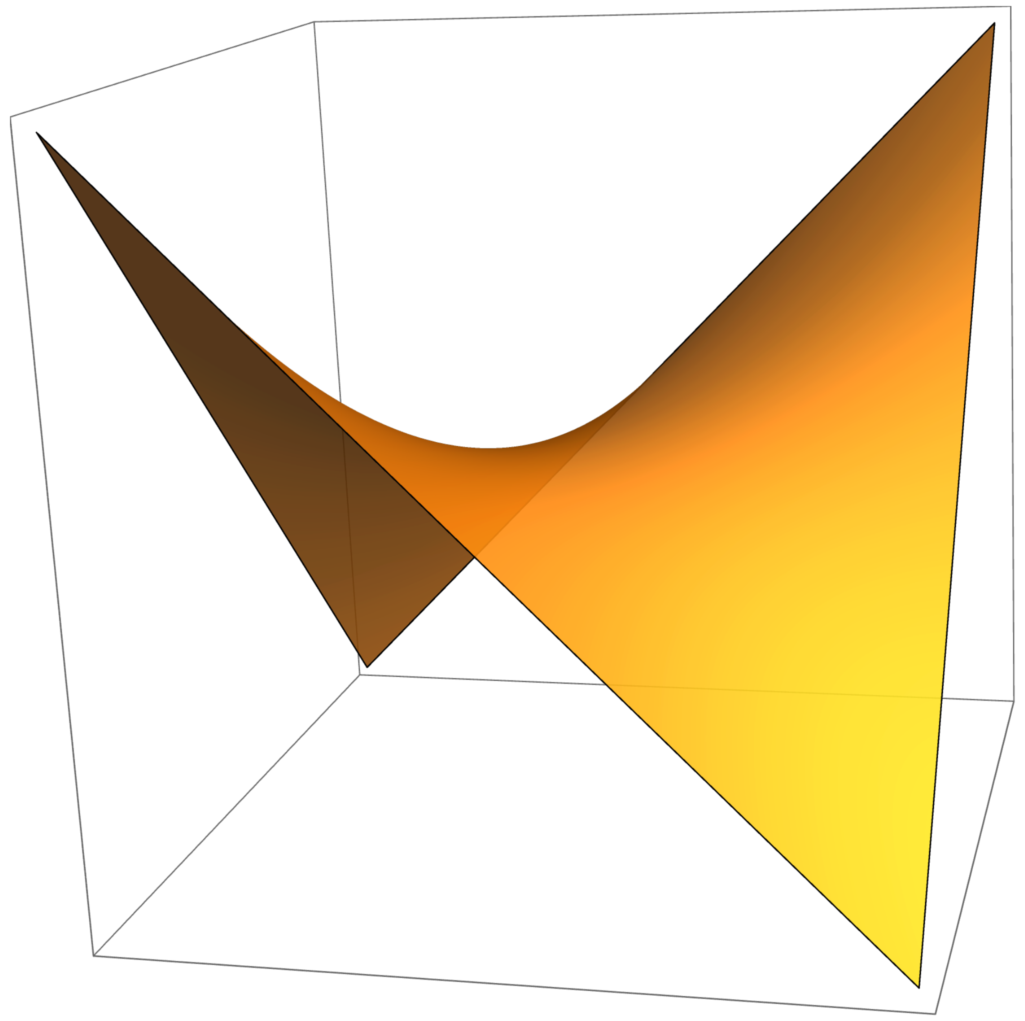
$$(t_1^1 t_2^2, t_1^1 t_2^1, t_1^2 t_2^3)$$



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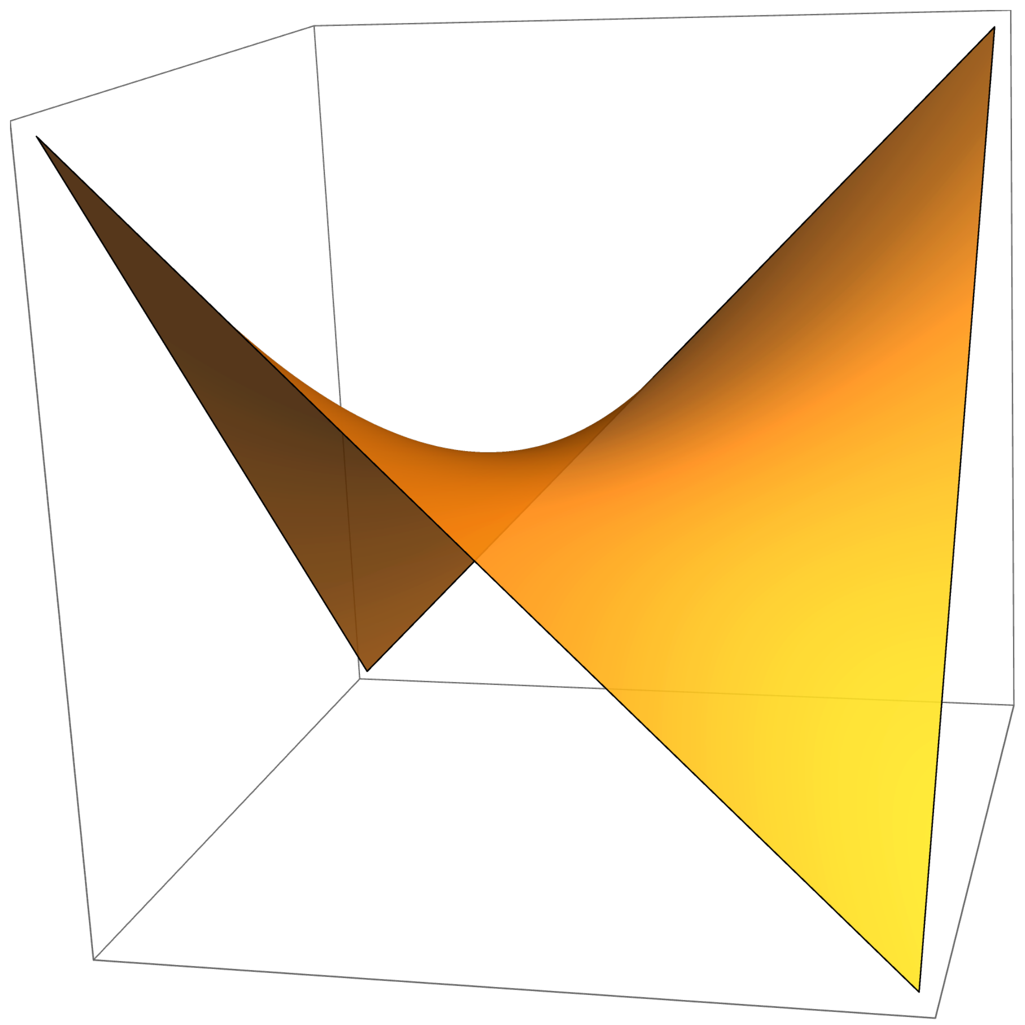
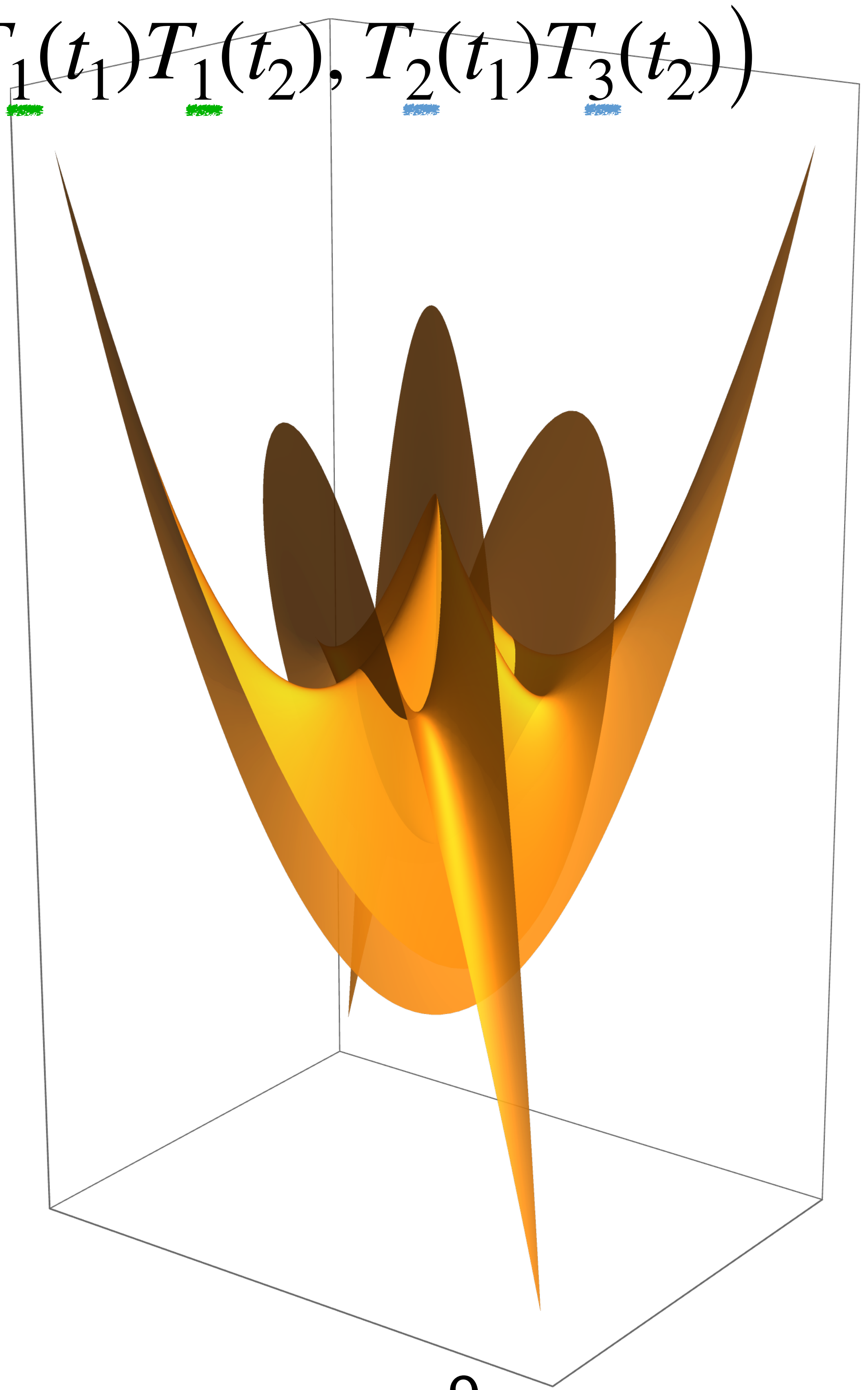
$$(t_1^1 t_2^2, t_1^1 t_2^1, t_1^2 t_2^3)$$

$$(T_1(t_1)T_2(t_2), T_1(t_1)T_1(t_2), T_2(t_1)T_3(t_2))$$





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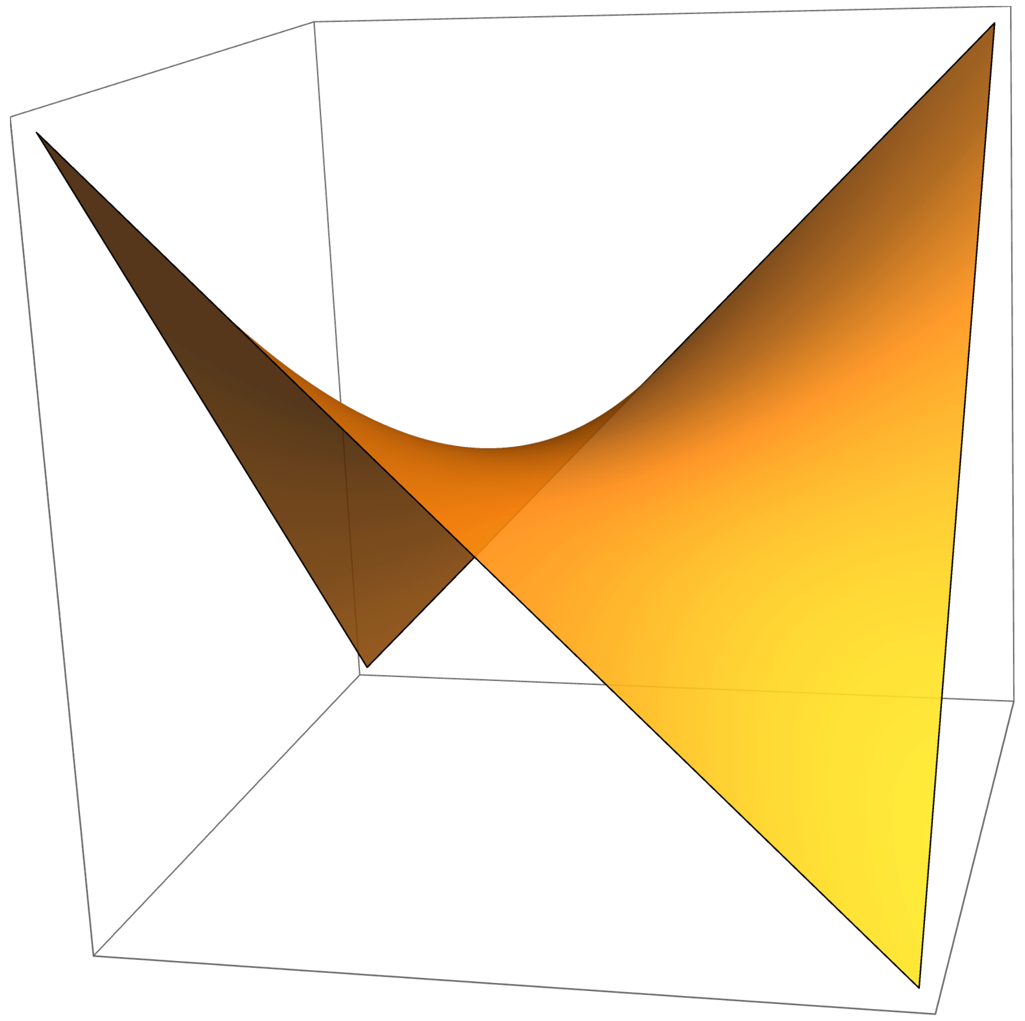
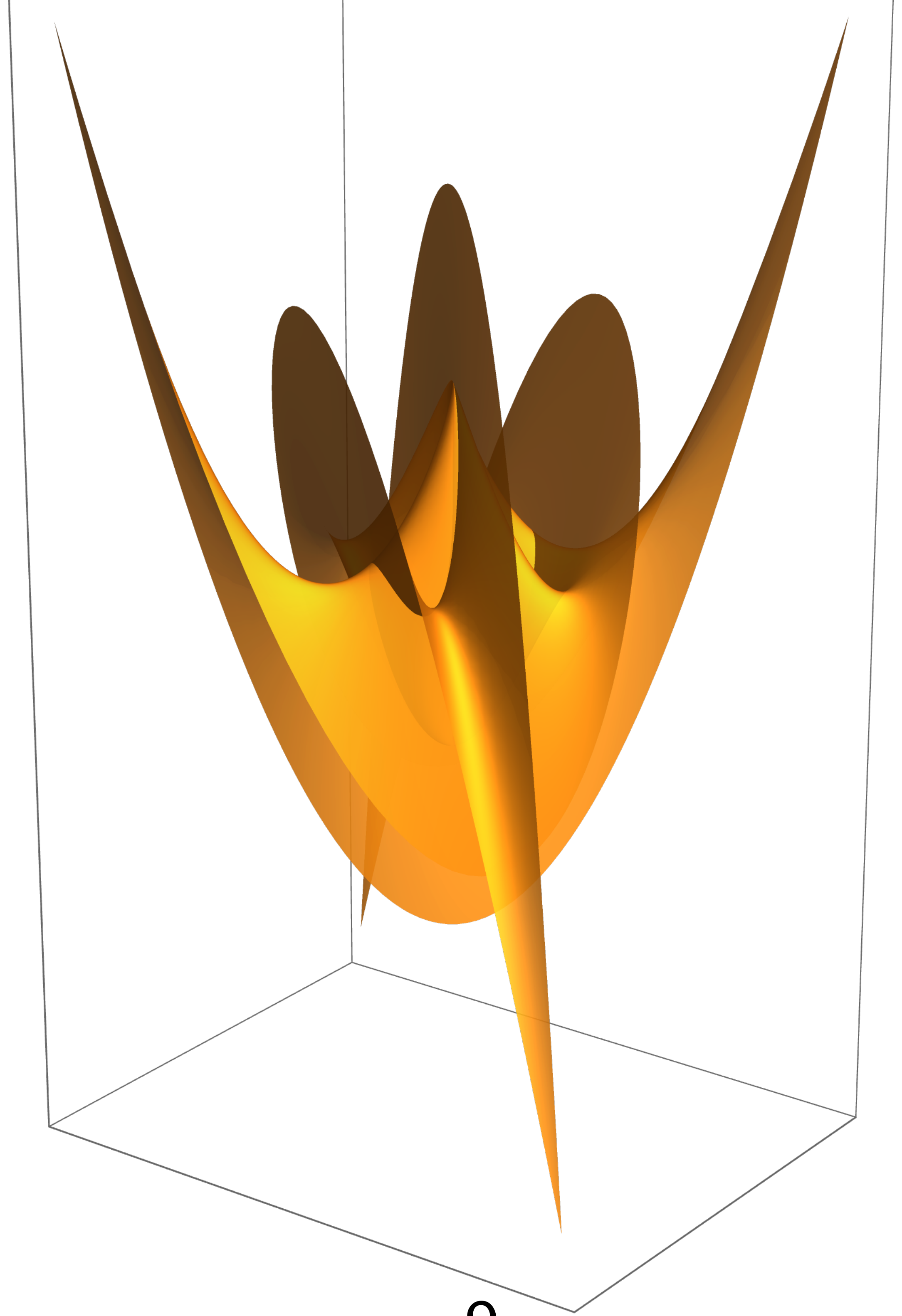
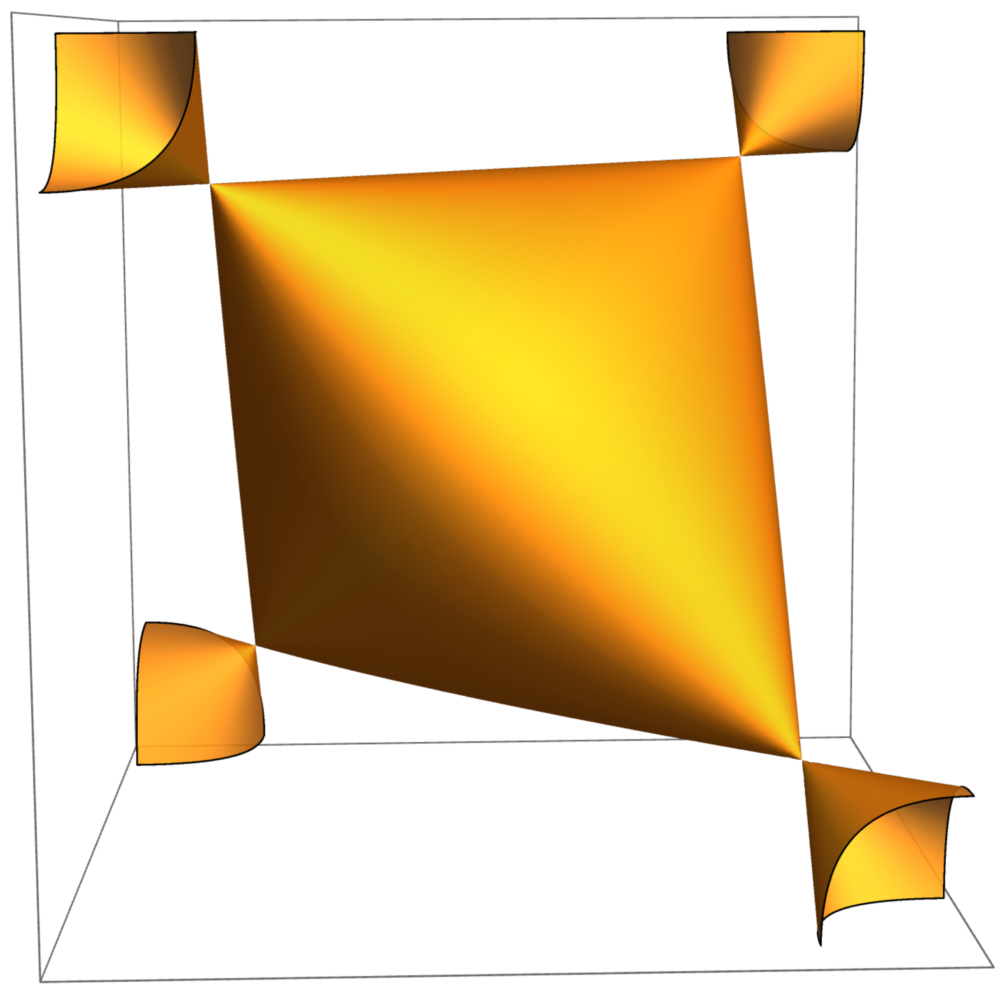
$\left(t_1^1 t_2^2, t_1^1 t_2^1, t_1^2 t_2^3\right)$

$\left(T_1(t_1)T_2(t_2), T_1(t_1)T_1(t_2), T_2(t_1)T_3(t_2)\right)$

$\left(\cos(1t_1 + 2t_2), \cos(1t_1 + 1t_2), \cos(2t_1 + 3t_2)\right)$



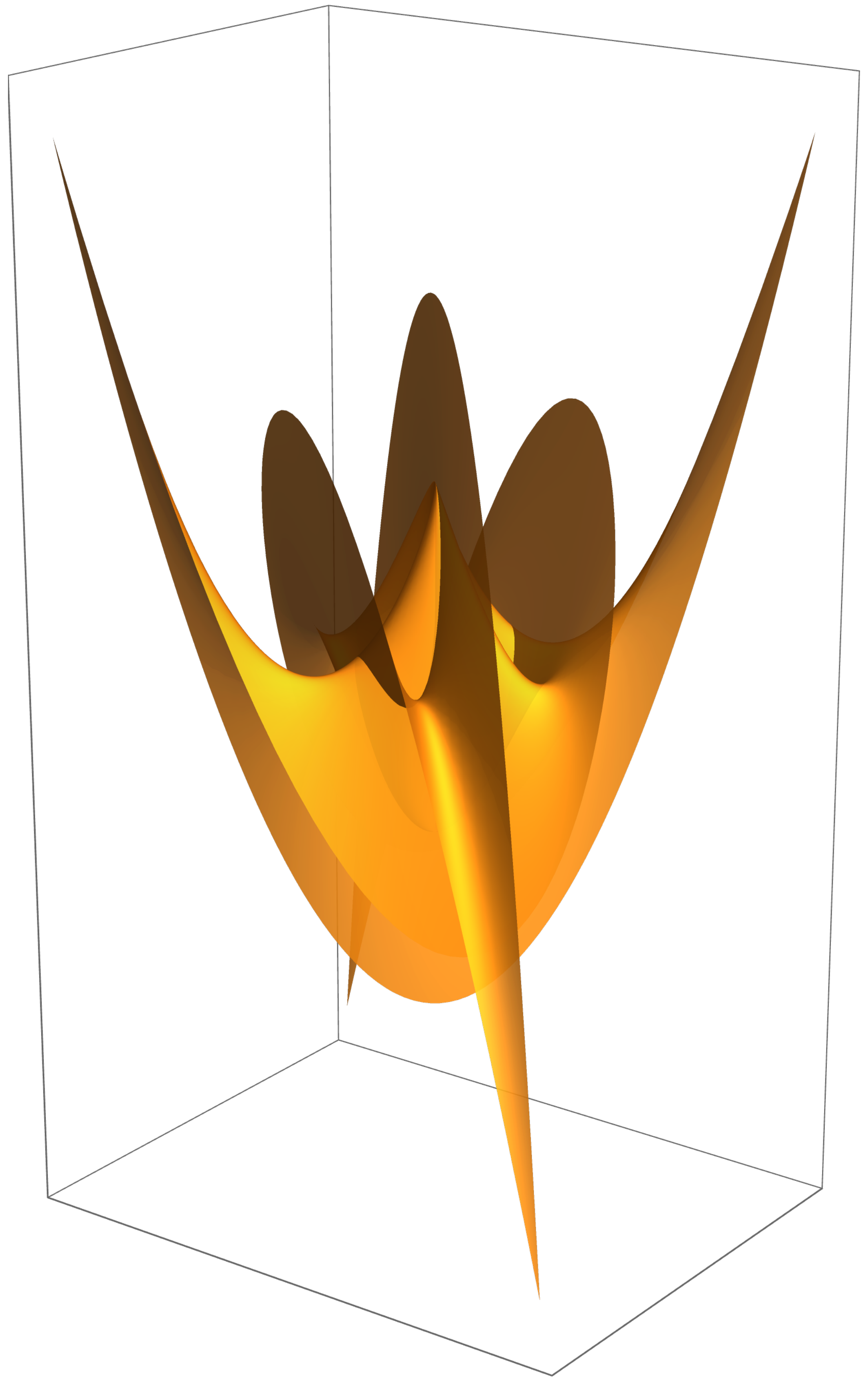




Tensor product Chebyshev varieties

$A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^{n \times s}$ full rank, $\phi_{A,\otimes} : \mathbb{C}^n \ni t \mapsto \left(T_{\alpha_{1,1}}(t_1) \cdot \ldots \cdot T_{\alpha_{1,n}}(t_n)\ ,\ \ldots\ ,\ T_{\alpha_{s,1}}(t_1) \cdot \ldots \cdot T_{\alpha_{s,n}}(t_n)\right) \in \mathbb{C}^s$

$X_{A,\otimes} = \overline{\text{image}(\phi_A)} \subset \mathbb{C}^s$



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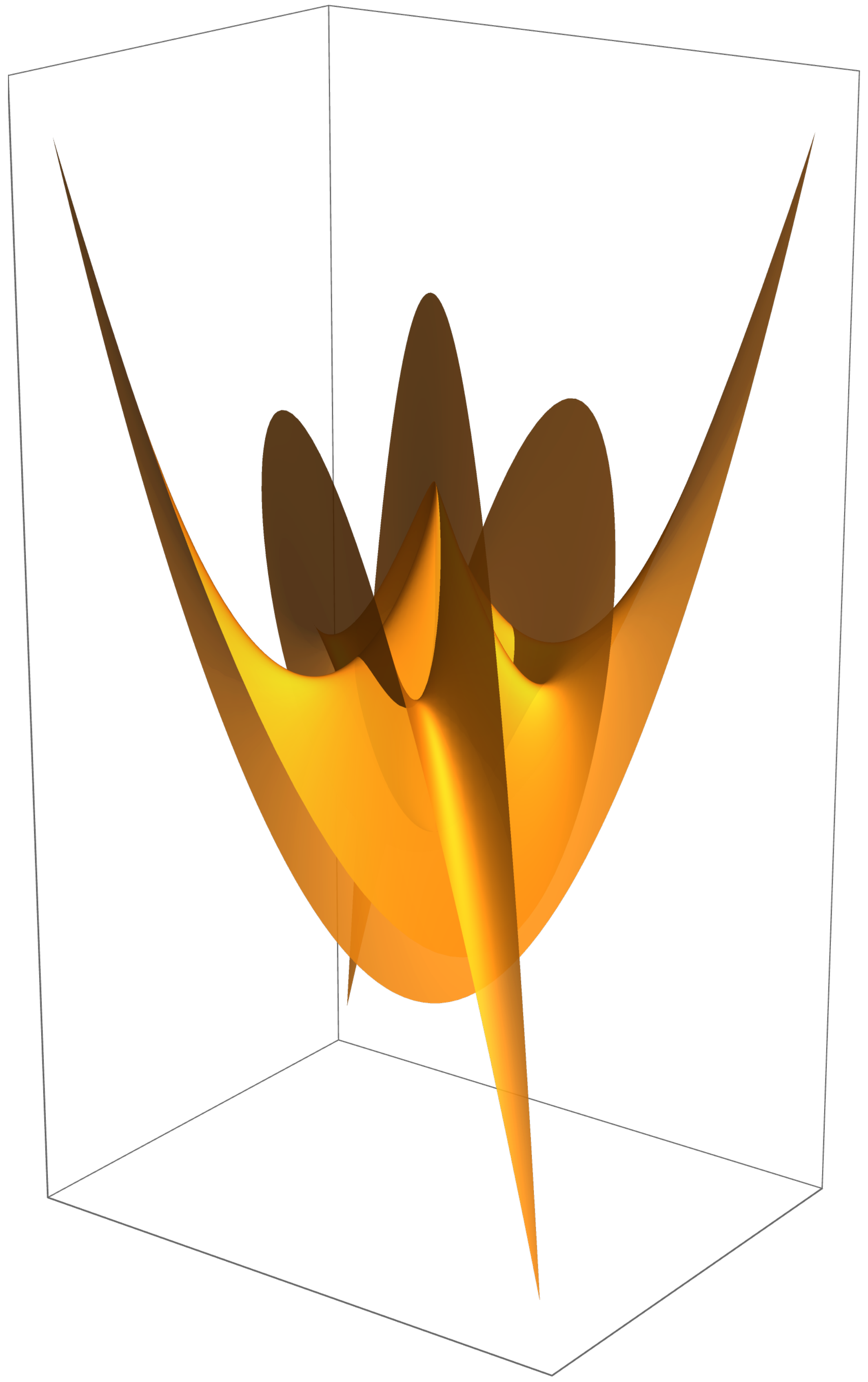

Tensor product Chebyshev varieties

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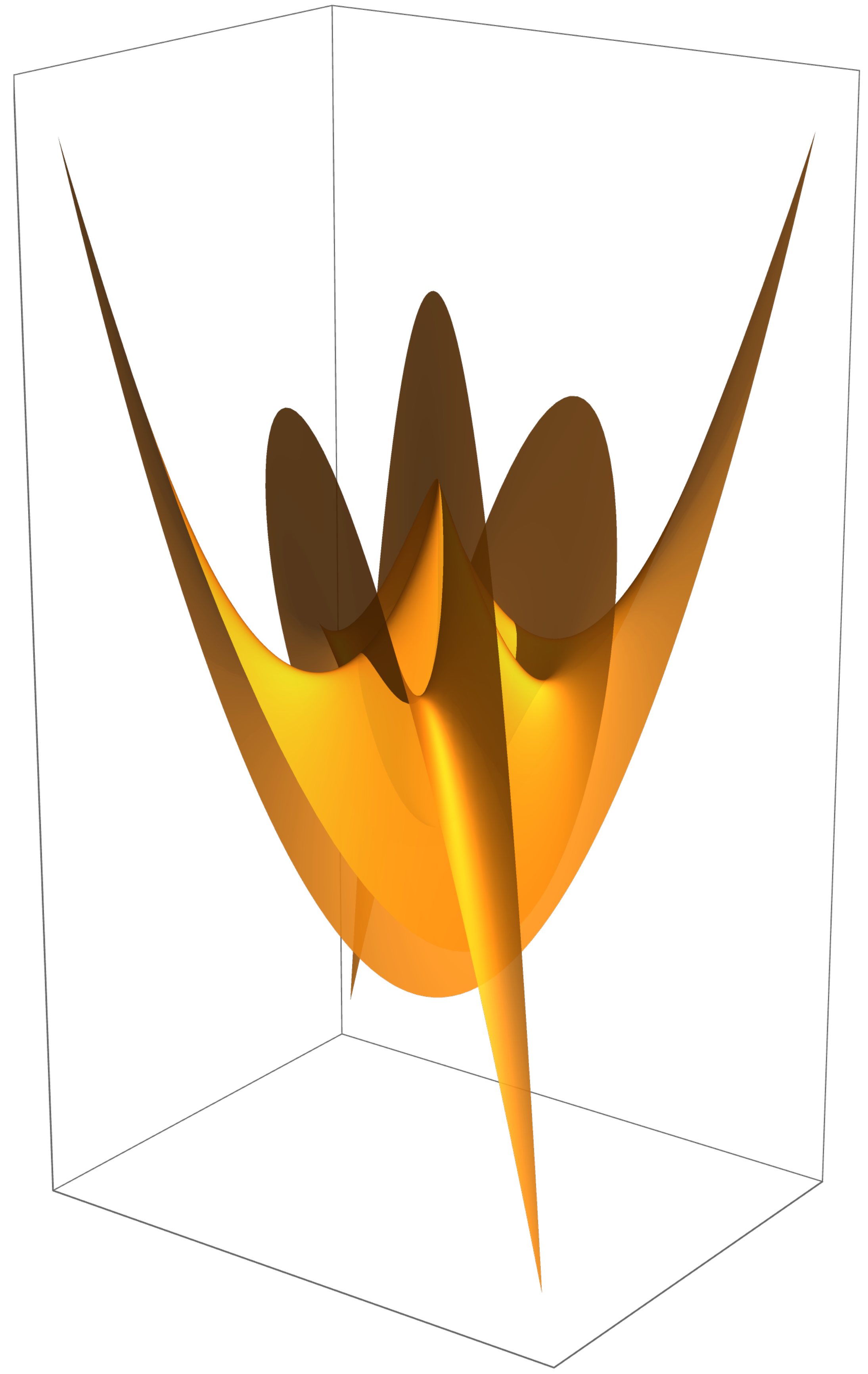
$A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^{n \times s}$ full rank, $\quad \phi_{A,\otimes} : \mathbb{C}^n \ni t \mapsto \left(T_{\alpha_{1,1}}(t_1) \cdot \ldots \cdot T_{\alpha_{1,n}}(t_n)\,,\ \ldots\ ,T_{\alpha_{s,1}}(t_1) \cdot \ldots \cdot T_{\alpha_{s,n}}(t_n)\right) \in \mathbb{C}^s$

$X_{A,\otimes} = \overline{\text{image}(\phi_A)} \subset \mathbb{C}^s$

$\dim X_{A,\otimes} = n$

$\deg X_{A,\otimes} = \ ?$



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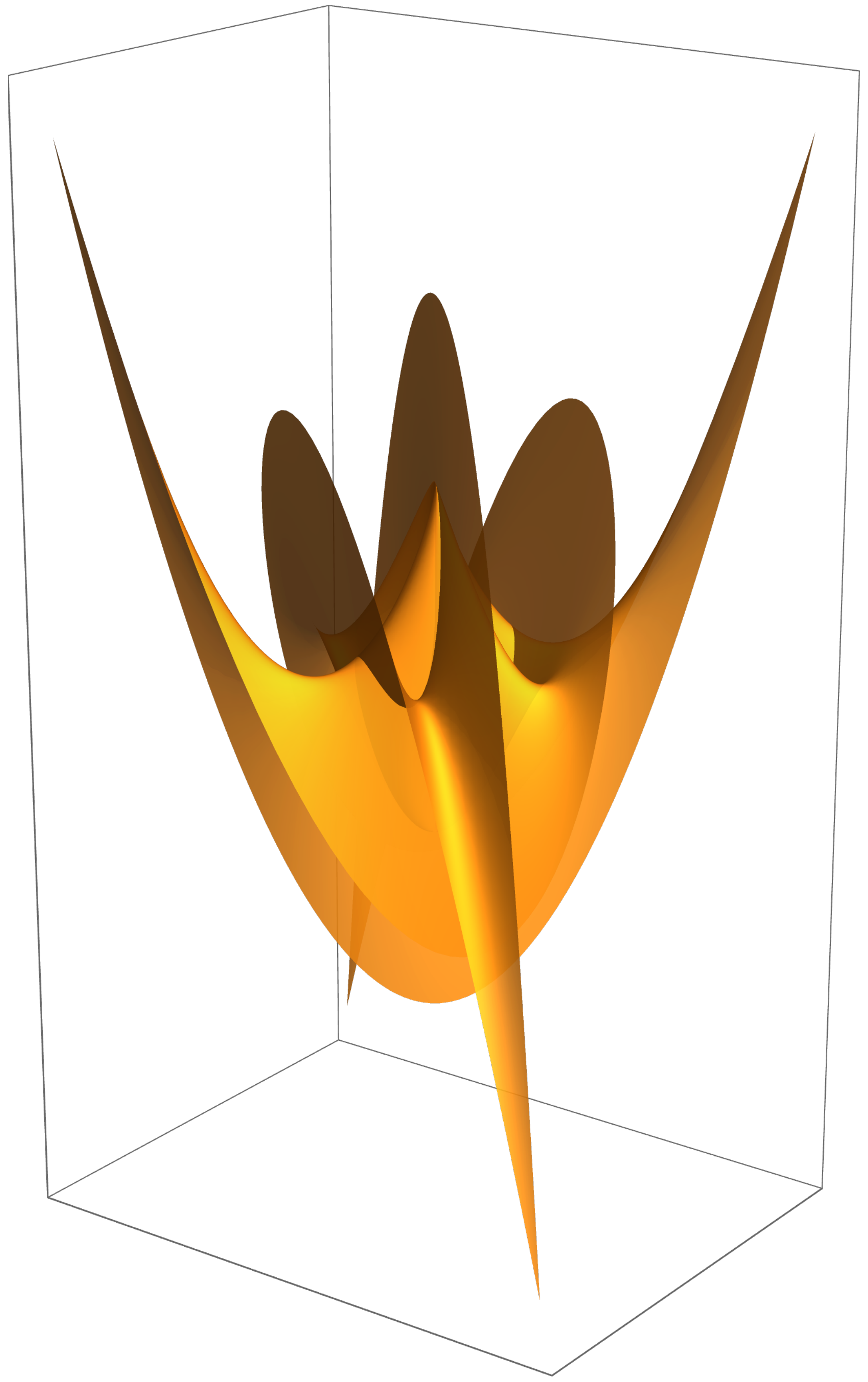

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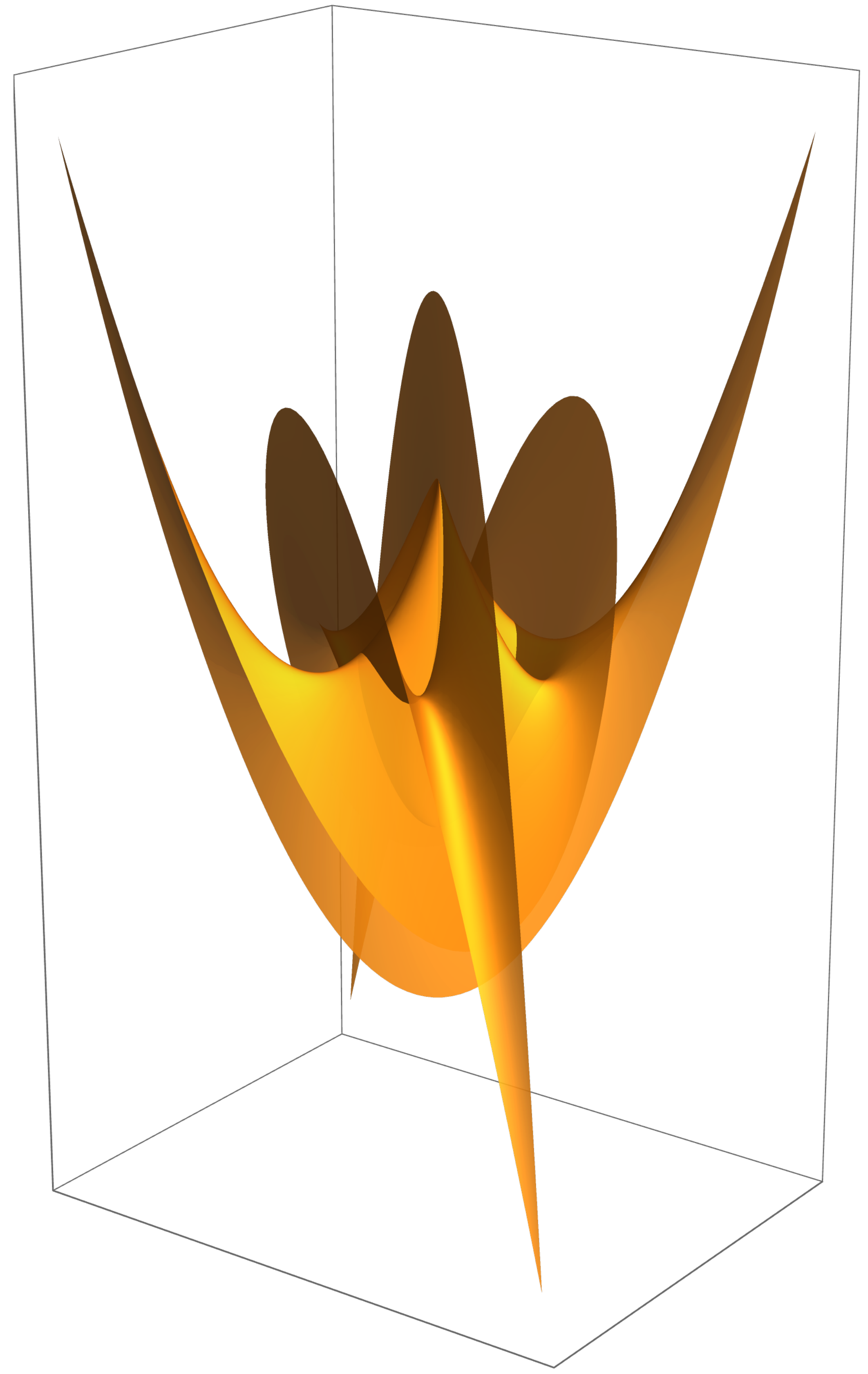

Under certain conditions it is the BKK prediction,
otherwise you can refine it exploiting combinatorial properties.

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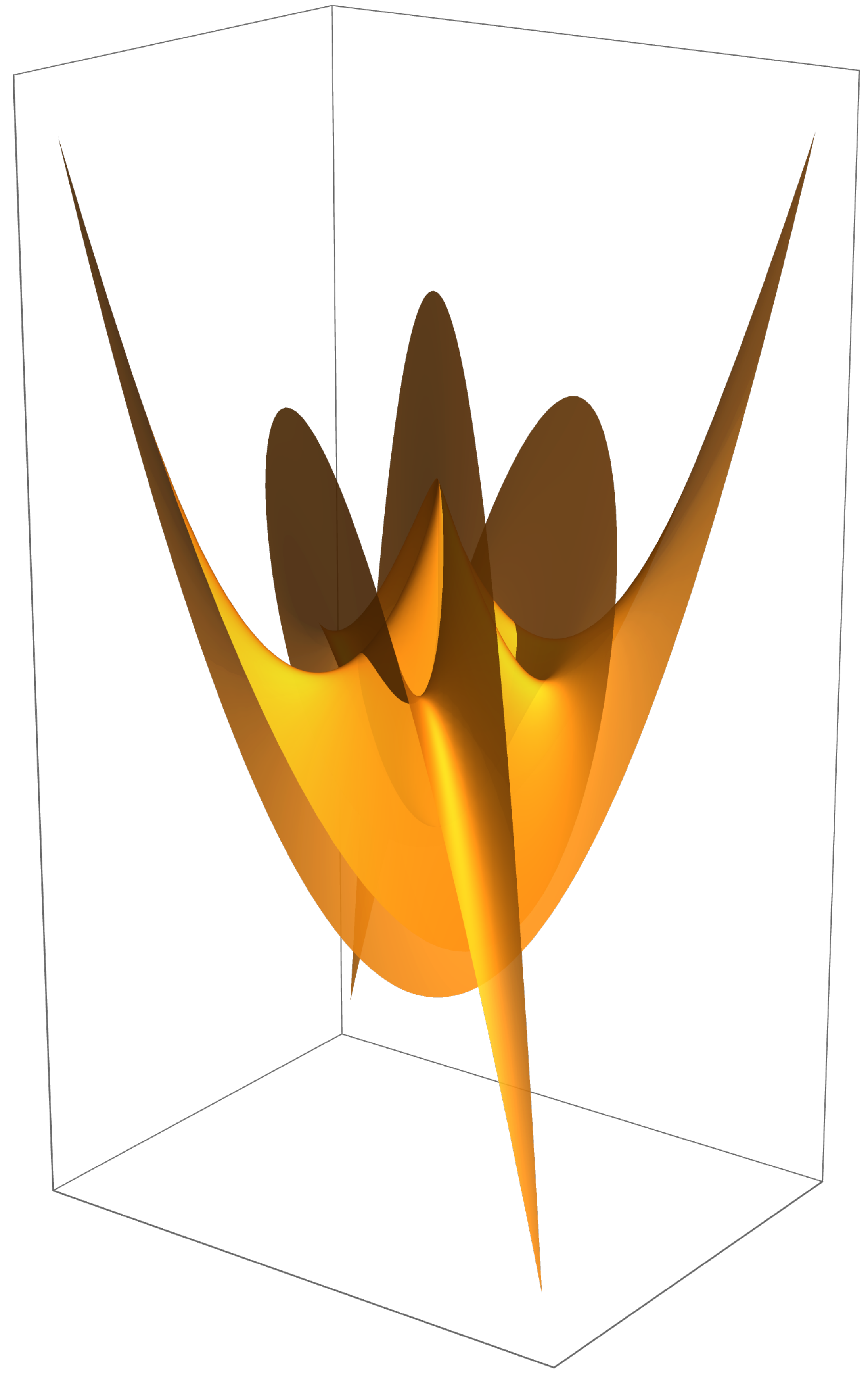

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Equations: open question

Trigonometric Chebyshev varieties

$A = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^{n \times s}$ full rank, $\phi_{A,\cos} : \mathbb{C}^n \ni t \mapsto \left(\cos(\alpha_1 \cdot t)\,,\ \ldots\ ,\ \cos(\alpha_s \cdot t)\right) \in \mathbb{C}^s$

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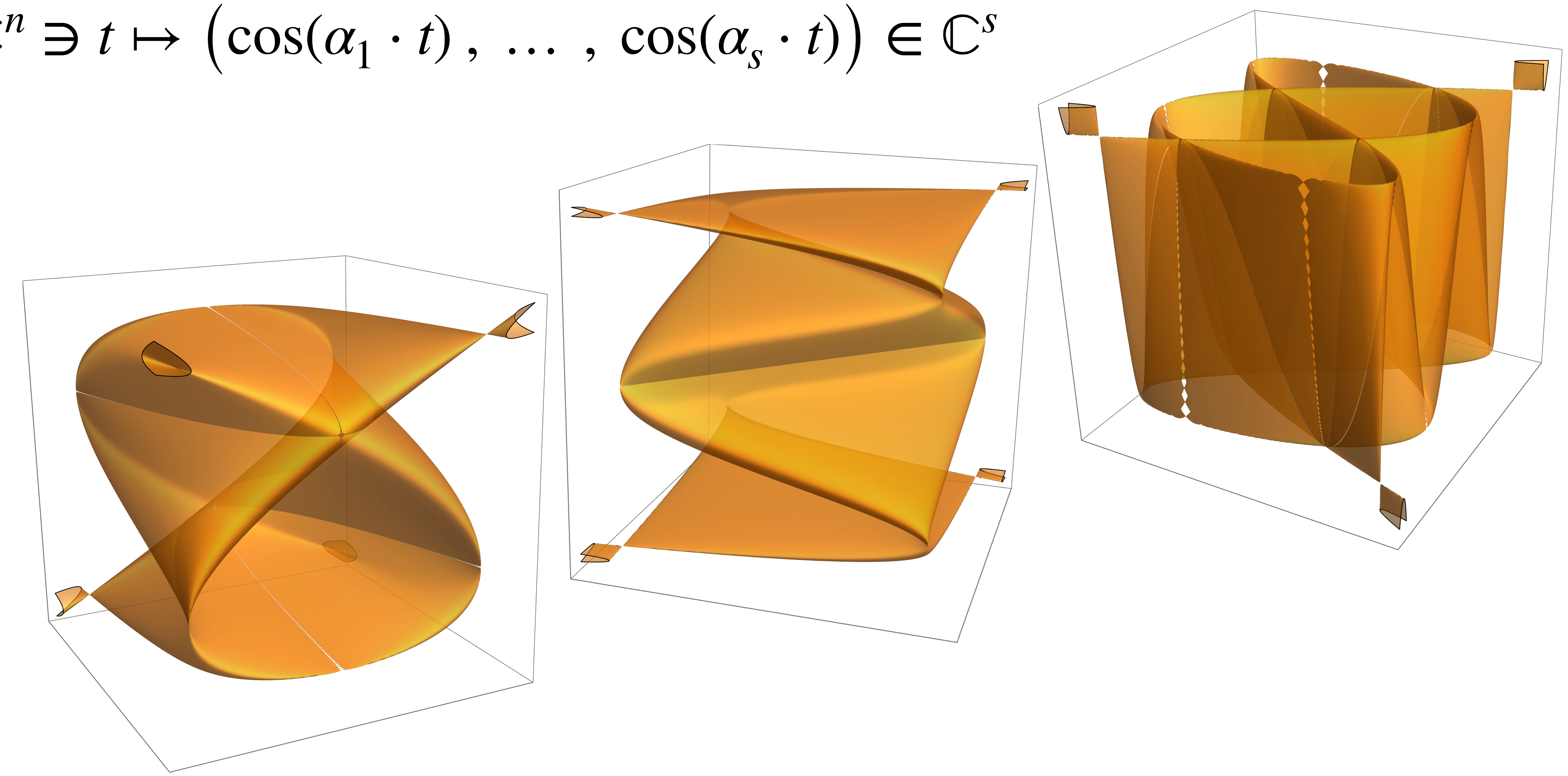

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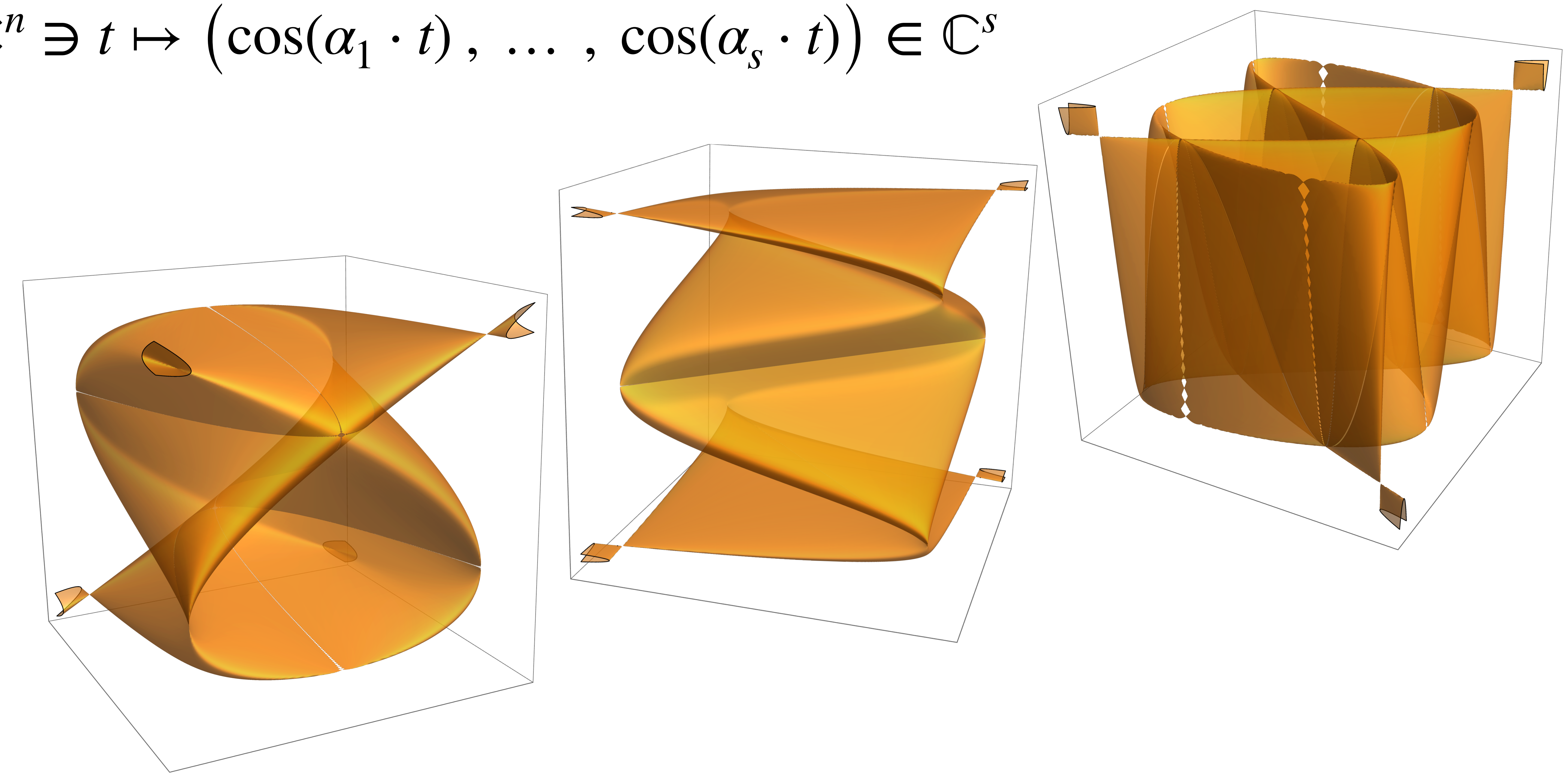

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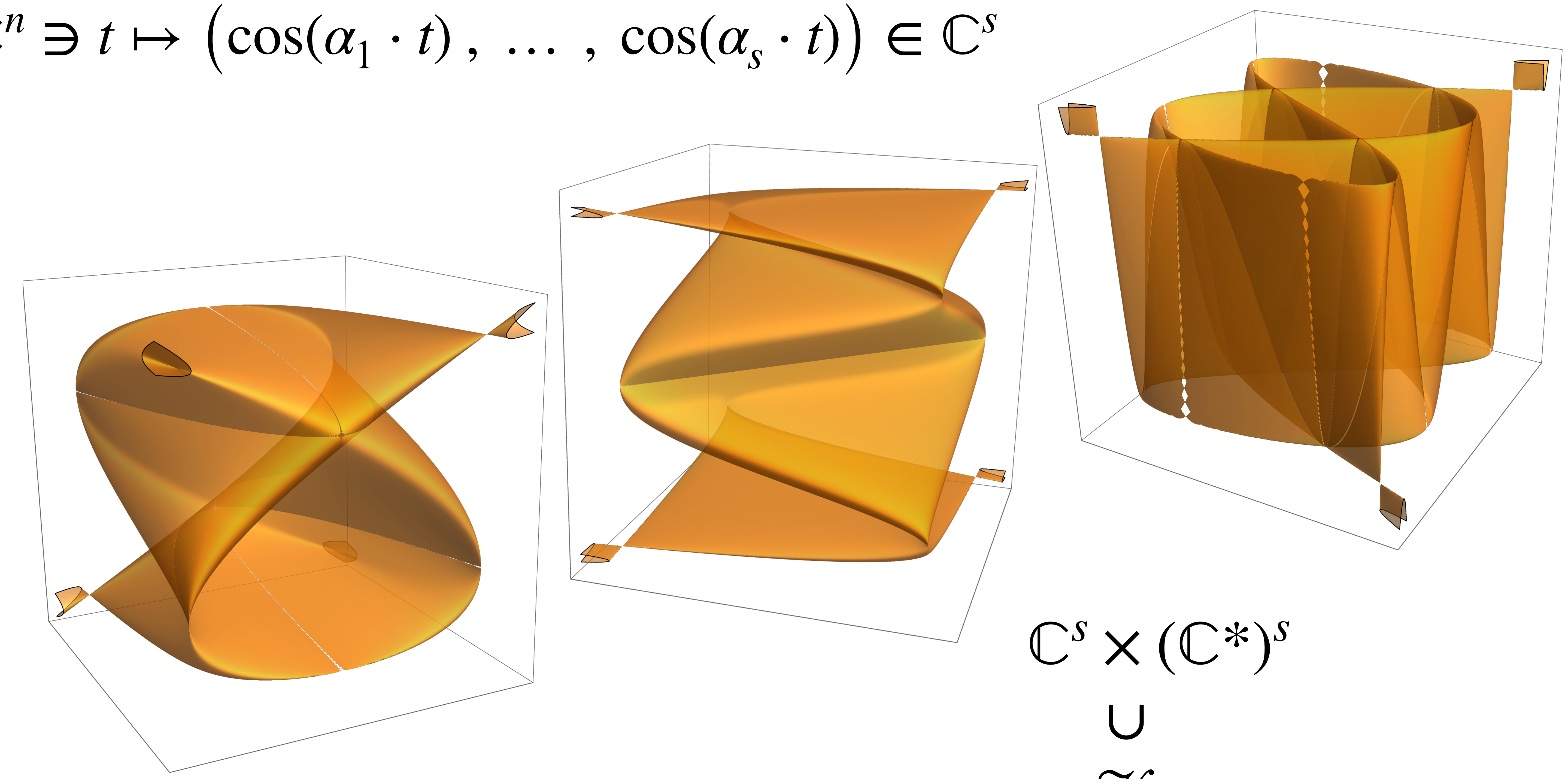
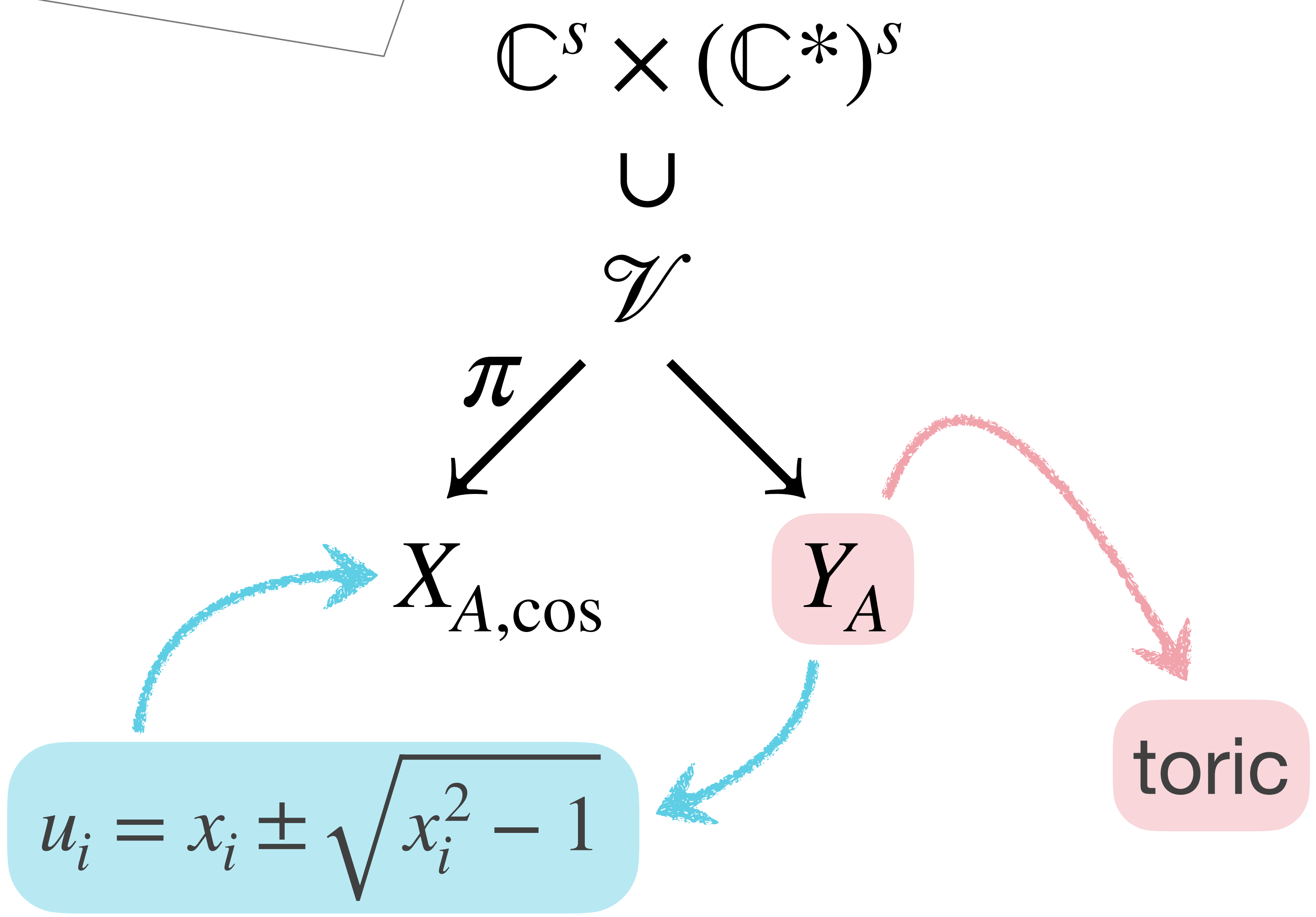


Theorem (Bel-Afia, M., Telen):
$X_{A,\cos} \subset \mathbb{C}^s$ is the Zariski closure of the projection of
$\mathscr{V} = \{(x,u) \in \mathbb{C}^s \times (\mathbb{C}^*)^s \mid u \in Y_A,\ u_i^2 - 2u_i x_i + 1 = 0 \ \text{ for } \ i = 1,\ldots,s\}$
onto $\mathbb{C}^s$. Moreover, $X_{A,\cos}$ is irreducible.



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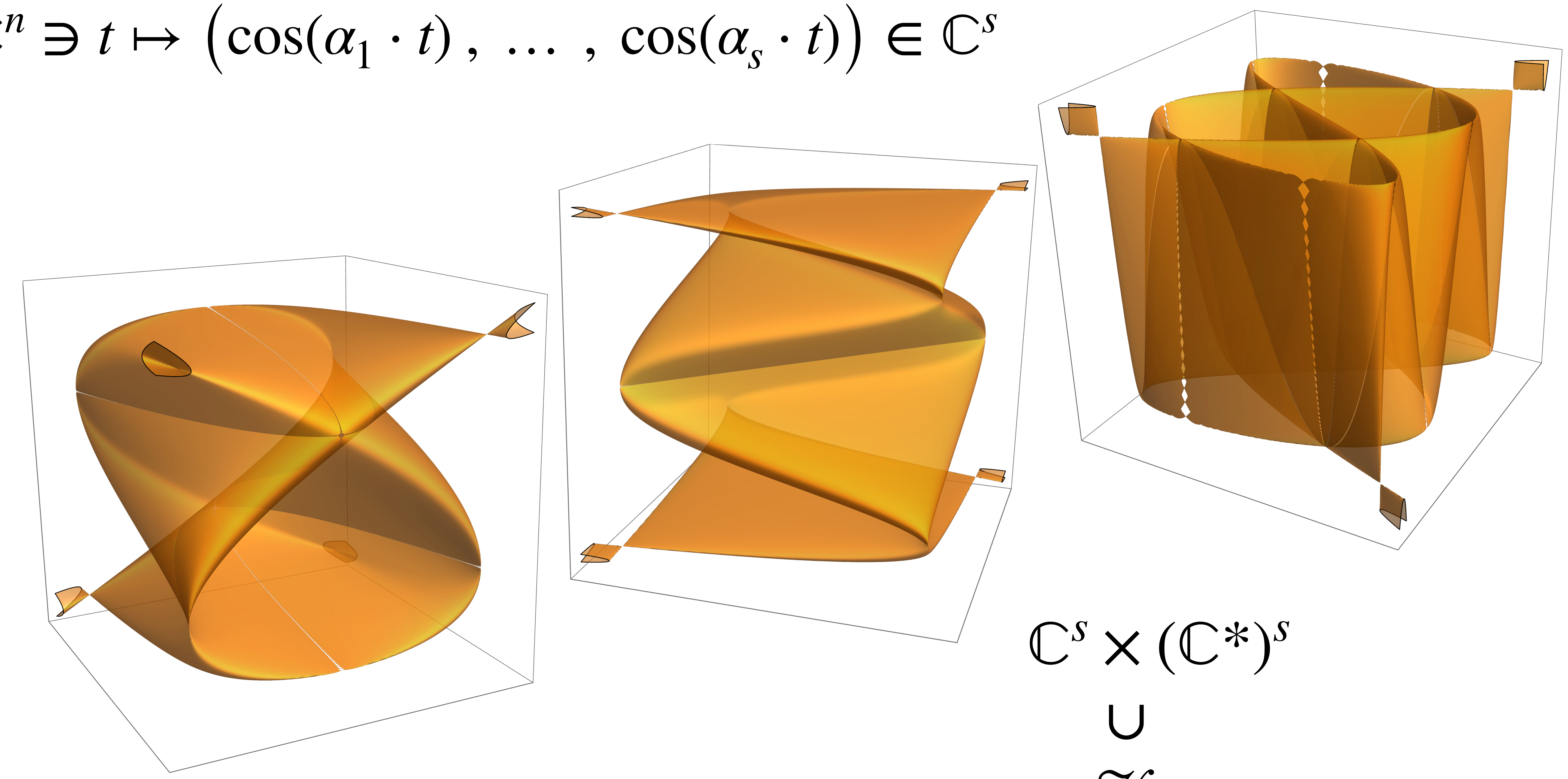
$$\deg X_{A,\cos} = \frac{\mathrm{vol}\big(\mathrm{conv}(A \cup -A)\big)}{\deg \pi \cdot \mathrm{ind}\, A}$$

Equations: ✓

Singularities: ✓



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$\mathbb{C}^s \times (\mathbb{C}^*)^s$
$\cup$
$\mathscr{V}$
$\pi \swarrow \quad \searrow$
$X_{A,\cos} \qquad Y_A$ → toric

$u_i = x_i \pm \sqrt{x_i^2 - 1}$

Experiment: solving systems

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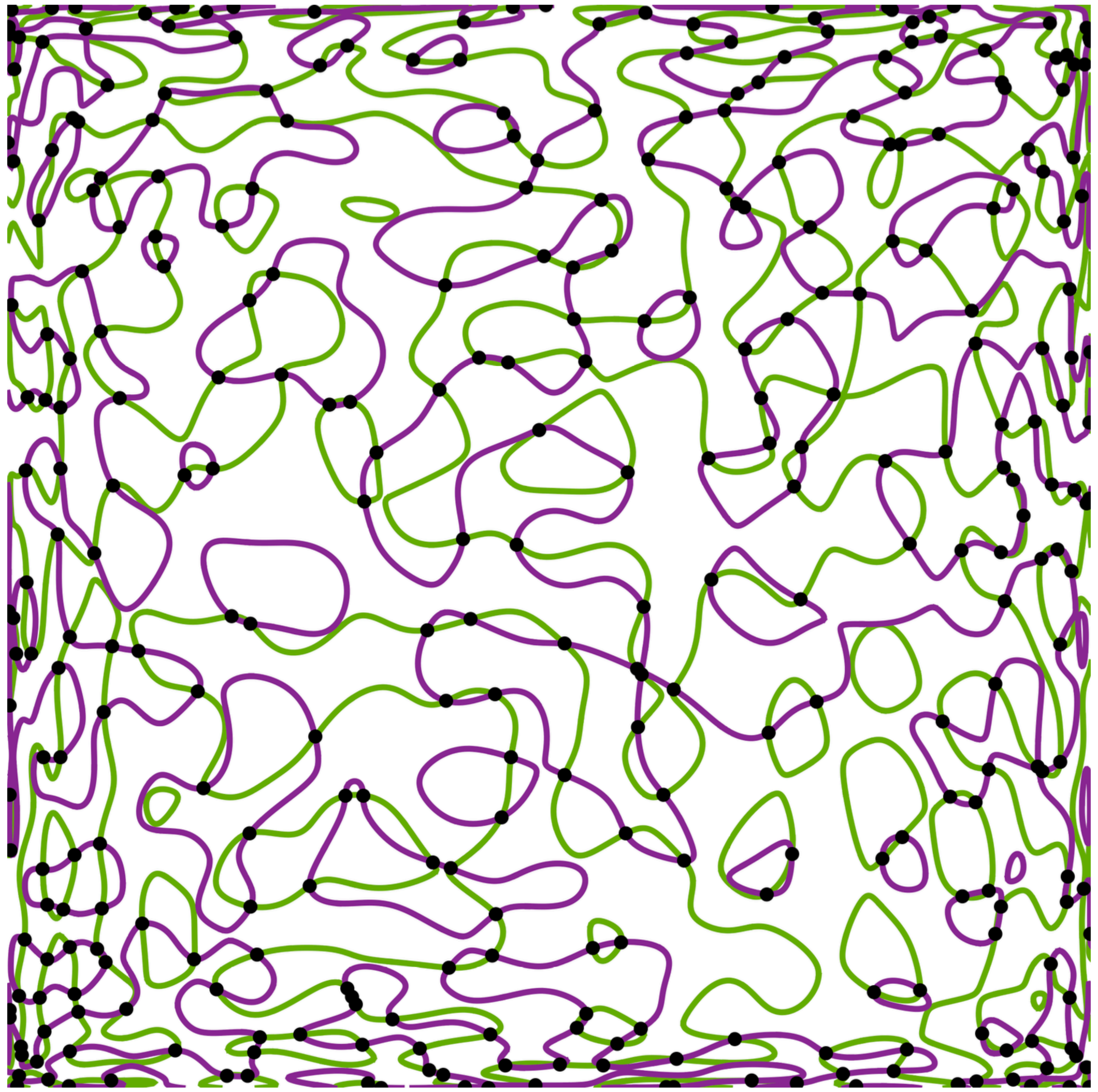
$$f_i(t) = c_{i,0} + \sum_{j=1}^{s} c_{i,j} T_{\alpha_j}(t) = 0, \quad i = 1,\dots,n, \quad t \in \mathbb{C}^n$$

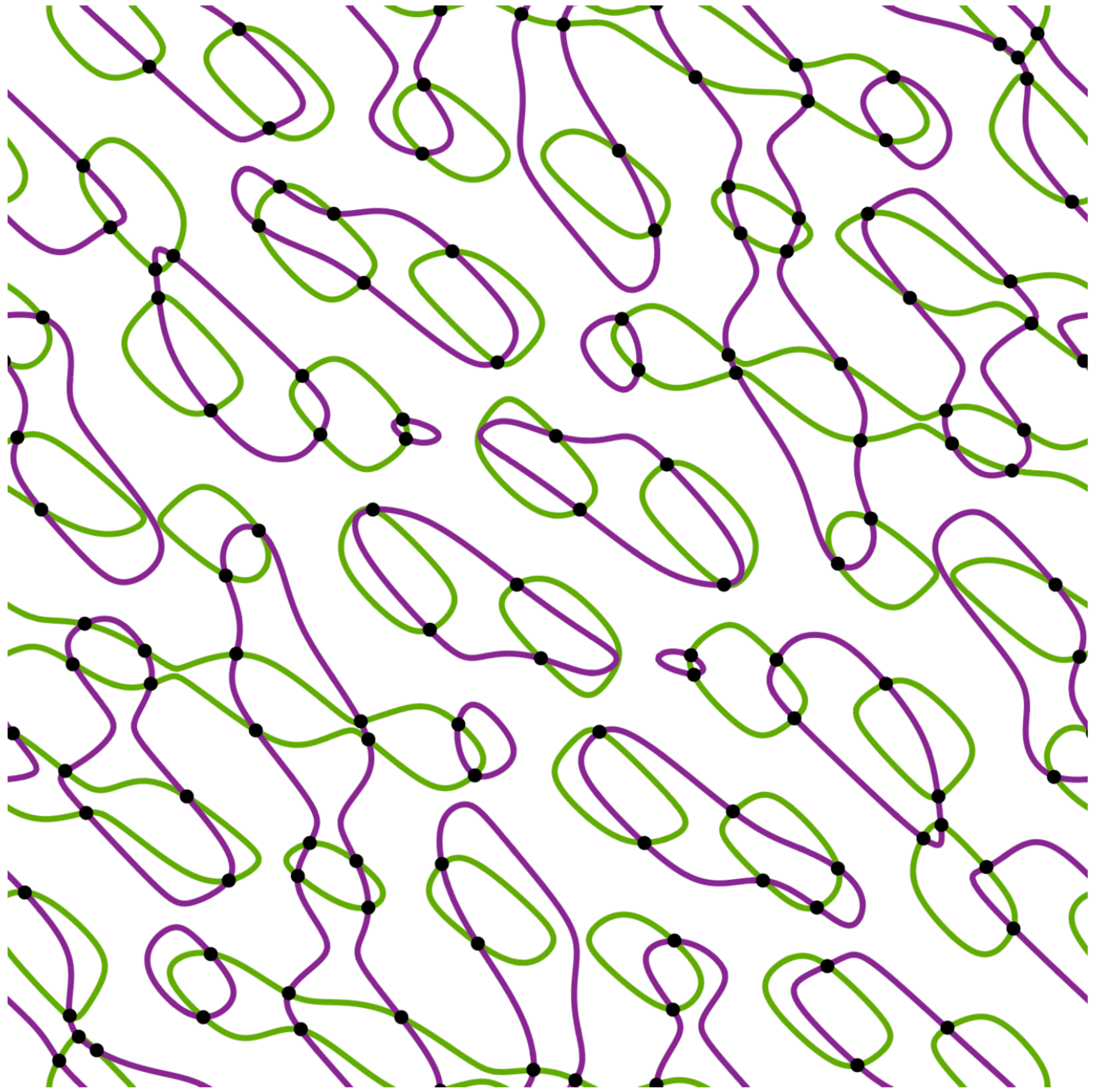
Eigenvalue algorithm



$n = 2$, Euclidean degree 30, $\deg X_A = 1396$, 382 real solutions.

$$f_i(t) = c_{i,0} + \sum_{j=1}^{s} c_{i,j} \cos(a_j \cdot t) = 0, \quad i = 1,\dots,n, \quad t \in \mathbb{C}^n$$

Monodromy



$n = 2$, $A = \begin{pmatrix} 4 & 4 & 6 & 7 & 9 & 2 \\ 8 & 4 & 1 & 2 & 6 & 7 \end{pmatrix}$, $\deg X_A = 129$, 258 complex solutions, 128 real solutions.

Generalizations

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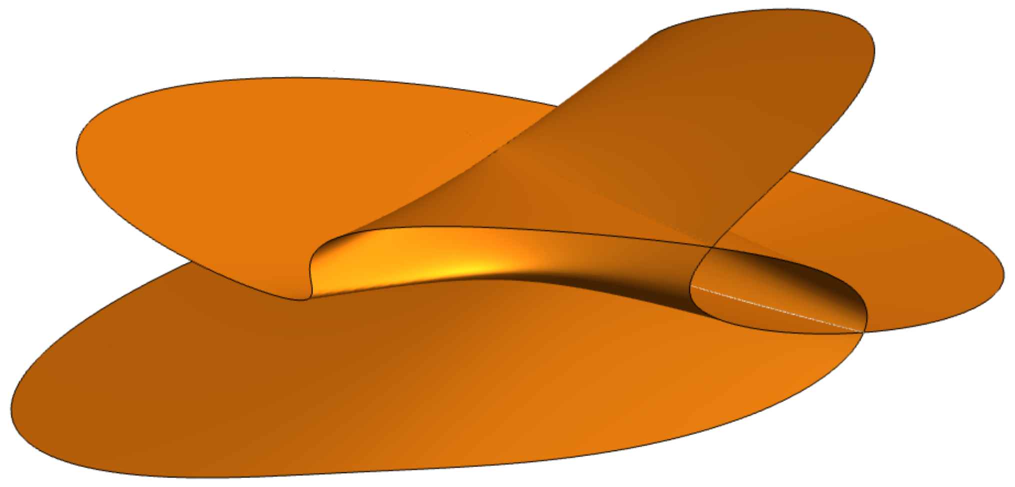
On multivariate Chebyshev polynomials and spectral approximations on triangles.
B.N. Ryland and H.Z. Munthe-Kaas (2010)

Sparse interpolation in terms of multivariate Chebyshev polynomials.
E. Hubert and M.F. Singer (2022)

For the root system $\mathscr{A}_2$ one has

$$T_{0,0} = 6,\ T_{1,0} = x,\ T_{0,1} = y,\ T_{1,1} = \frac{1}{4}xy - 3,\ \ldots$$



$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

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Analogously for the basis of harmonic polynomials

Is (and how) the Real structure of the associated varieties rich?

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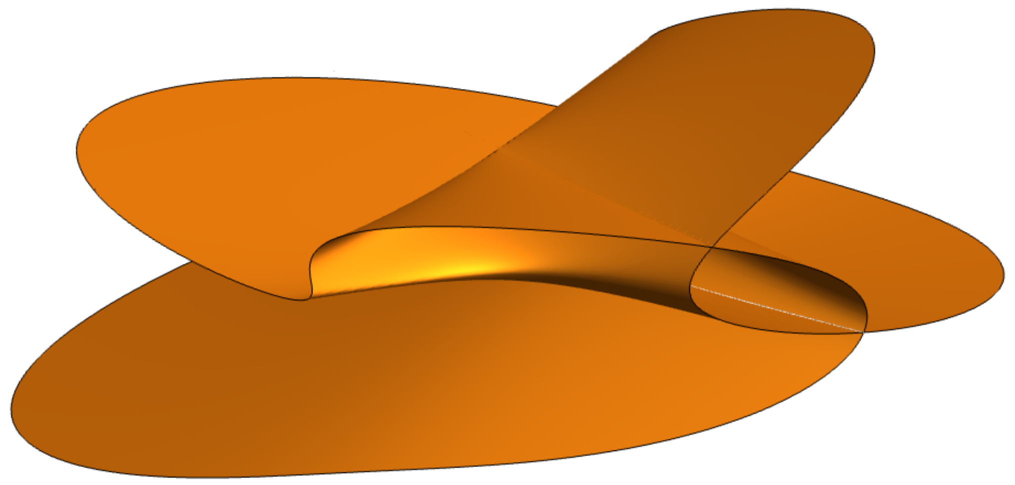

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On fully real eigenconfigurations of tensors.
K. Kozhasov (2018)

Real lines on random cubic surfaces.
R. Ait El Manssour, M. Belotti, CM (2021)

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monomials : Toric varieties = Chebyshev polynomials : Chebyshev varieties

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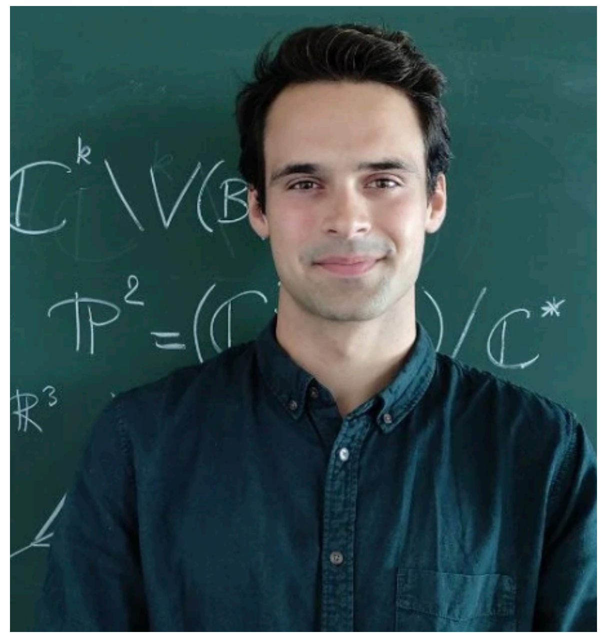
[arXiv:2401.12140]



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monomials : Toric varieties = Chebyshev polynomials : Chebyshev varieties

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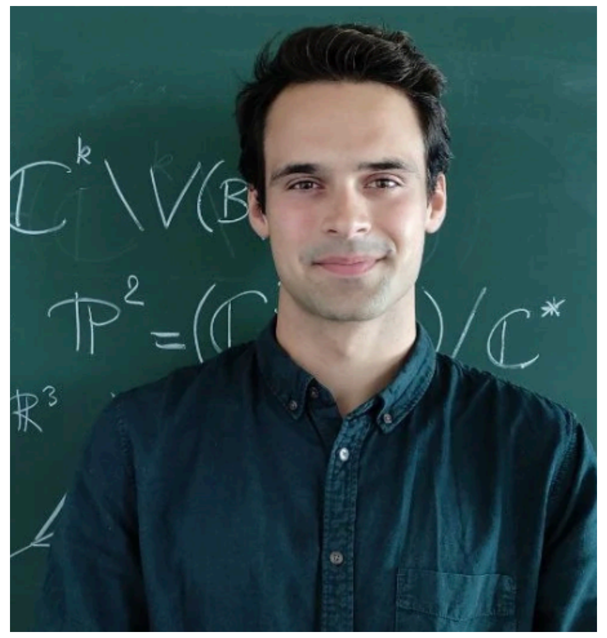

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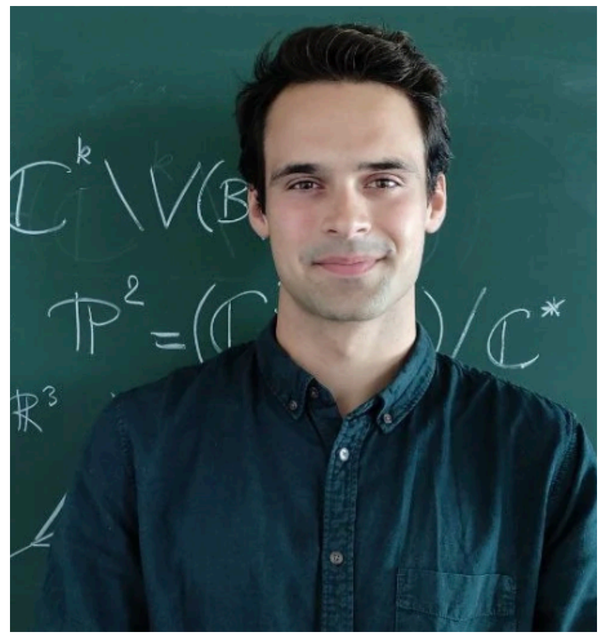

Thank you!

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