Degenerations and smoothings of Fano varieties: computational aspects

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Fano varieties

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Example: Projective space

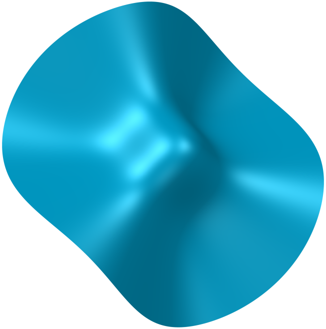
$$\mathbb{P}^3 = (\mathbb{C}^4 \setminus \{0\})/\sim$$

Example: Smooth cubic surface

$$C = \{F_3 = 0\} \subset \mathbb{P}^3$$

where

$$F_3 = X^3 + Y^3 + Z^3 - W^3$$



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For Fano surfaces, the number has been known classically.

Theorem

There are essentially only 10 families of del Pezzo surfaces, i.e., $\pi\colon \mathcal{X} \to M$ can be chosen s.t. M has 10 irr. conn. components.

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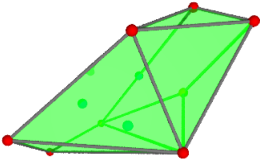
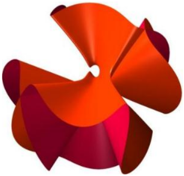
Goal

Develop new methods to construct Fano varieties as a step toward a complete list in higher dimensions.

Starting point: Gorenstein toric Fano varieties





reflexive polytope $\Delta \subset \mathbb{R}^d$ $\rightsquigarrow$ toric variety $X(\Delta)$, Gorenstein and Fano

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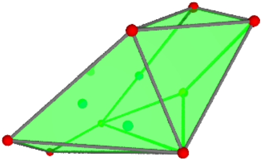

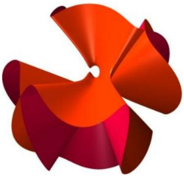

reflexive polytope $\Delta \subset \mathbb{R}^d$ $\rightsquigarrow$ toric variety $X(\Delta)$, Gorenstein and Fano

In any dimension d, there are only finitely many reflexive polytopes.

dim	#	smooth #
1	1	1
2	16	5
3	4,319	18
4	473,800,776	124

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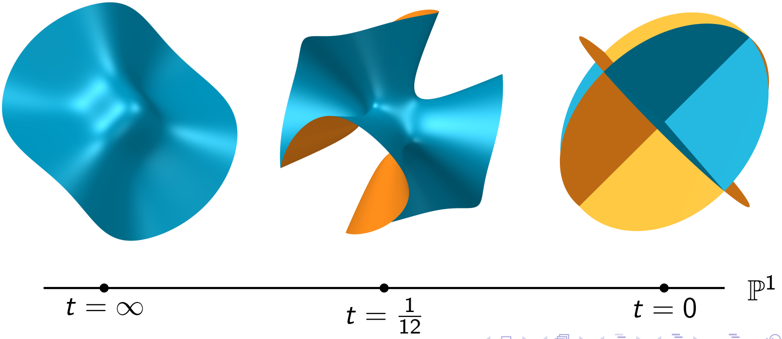
$$\mathcal{X} = \{XYZ - t \cdot (X^3 + Y^3 + Z^3 - W^3) = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1$$



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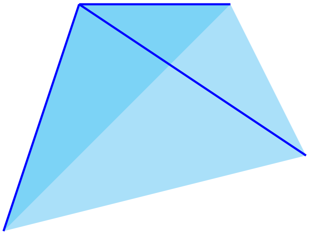
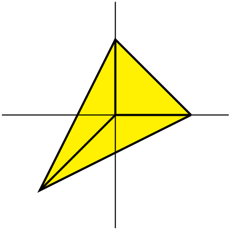
algebraic geometry



central fiber $X_0 \subset \mathbb{P}^3$

affine geometry



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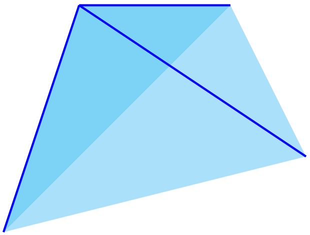
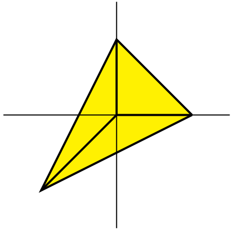


central fiber $X_0 \subset \mathbb{P}^3$

degenerate Fano variety $X_0(\Delta)$

affine geometry



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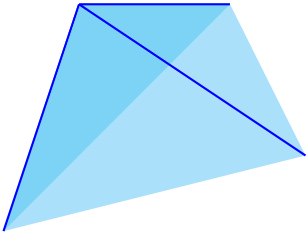	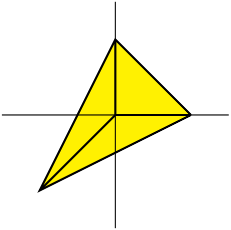
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degenerate Fano variety $X_0(\Delta)$	reflexive polytope Δ with central subdivision $\mathscr{P}_\Delta$

A reflexive polytope Δ gives rise to a degenerate Fano var. $X_0(\Delta)$.

How can we construct smooth Fano varieties?

Aim: Recognize $X_0(\Delta)$ as degenerate fiber of a degenerating family $f\colon X \to \mathbb{A}^1$ of smooth Fano varieties.

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Strategy

Construct, order by order, infinitesimal thickenings

$$f_k\colon X_k(\Delta) \to S_k = \operatorname{Spec} \mathbb{C}[t]/(t^{k+1}).$$

$$\begin{array}{ccccccccccccc}
X_0(\Delta) & \to & X_1 & \to & X_2 & \to & \dots & \to & \mathfrak{X} & \to & X & \leftarrow & X_\eta \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\
S_0 & \to & S_1 & \to & S_2 & \to & \dots & \to & \operatorname{Spf} \mathbb{C}[\![t]\!] & \to & \operatorname{Spec} \mathbb{C}[\![t]\!] & \leftarrow & \operatorname{Spec} \mathbb{C}(\!(t)\!)
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Method

Use logarithmic deformation theory:

(1) log scheme = scheme + extra structure

(2) Log smooth deformations are locally unique.

(3) Log smooth deformations approach a smoothing in the limit.

How can we construct smooth Fano varieties?

"Dream Recipe"

(1) Construct degenerate Fano variety $X_0(\Delta)$.

(2) Endow $X_0(\Delta)$ with a log structure to obtain log smooth morphism

$$X_0^\dagger(\Delta) \to S_0^\dagger, \qquad S_0^\dagger := \mathrm{Spec}(\mathbb{N} \xrightarrow{1 \mapsto 0} \mathbb{C}).$$

(3) Show existence of infinitesimal log smooth deformations

$$X_k^\dagger(\Delta) \to S_k^\dagger, \qquad S_k^\dagger := \mathrm{Spec}(\mathbb{N} \xrightarrow{1 \mapsto t} \mathbb{C}[t]/(t^{k+1})),$$

up to any order.

(4) Obtain smoothing $X \to S$ as limit.

Log singularities

Log deformation theory works well for log smooth spaces, but ...

... our example $f\colon \mathcal{X} \to \mathbb{P}^1$ is not log smooth!

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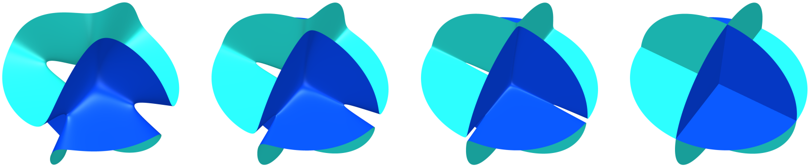

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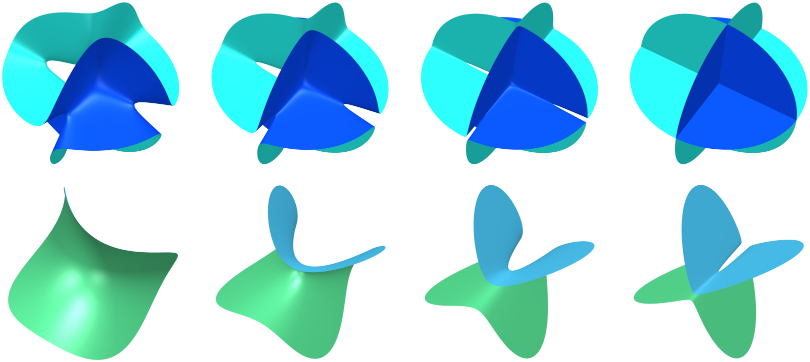

Generically log smooth families

Definition

A generically log smooth family consists of:

(a) a flat and separated morphism of finite presentation $f\colon X \to S$ whose fibers satisfy Serre's condition (S_2) and are geometrically reduced;

(b) an open subset $j\colon U \xrightarrow{\subseteq} X$ whose complement $Z = X \setminus U$ has relative codimension ≥ 2;

(c) the structure of a log smooth and saturated morphism

$$f \circ j\colon U^\dagger \to S^\dagger$$

of log schemes.

Generically log smooth families

Definition

A class of log singularities $\mathscr{C}$ is a set of gls families over $\operatorname{Spec}\mathbb{C}[\![t]\!]$, the set of local models. A gls family $f_k\colon X_k \to S_k$ is of class $\mathscr{C}$ if it is, locally in the étale topology, isomorphic to the base change of a family in $\mathscr{C}$, i.e., of a local model.

"Definition"

A class of log singularities $\mathscr{C}$ is mild if certain technical conditions are met. For example, the Hodge–de Rham spectral sequence

$$H^q(X_0, \mathcal{A}^p_{X_0/S_0}) \Rightarrow \mathbb{H}^{p+q}(X_0, \mathcal{A}^\bullet_{X_0/S_0})$$

degenerates at E_1. Here, $\mathcal{A}^\bullet$ is some log de Rham complex.

The logarithmic Bogomolov–Tian–Todorov theorem

Theorem (Chan–Leung–Ma 2019, F–Filip–Ruddat 2019, F 2019, F–Petracci 2021, F 2023)

Let $\mathscr{C}$ be a mild class of log singularities. Let $f_0 \colon X_0 \to S_0$ be a proper gls family of class $\mathscr{C}$. Assume that f_0 is log Calabi–Yau. Then the logarithmic deformation functor

$$\mathrm{LD}^{\mathscr{C}}_{X_0/S_0} \colon \mathrm{Art}_{\mathbb{C}[\![t]\!]} \to \mathrm{Set}$$

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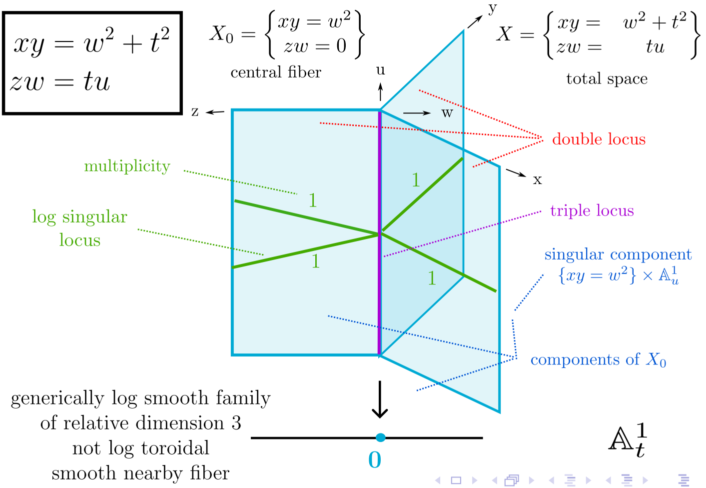
Corollary

In the situation of the theorem, there is a (partial) smoothing $X_\eta/\mathbb{C}(\!(t)\!)$ of X_0.

Finding the right class of log singularities for $X_0(\Delta)$

Candidate from some construction of log structures on $X_0(\Delta)$:



Finding the right class of log singularities for $X_0(\Delta)$

It is difficult to determine if a given class of log sing. $\mathscr{C}$ is mild.

Fact

If $\mathscr{C}$ is a mild class of log singularities, and if $f \colon X \to \operatorname{Spec} \mathbb{C}[\![t]\!]$ is proper and of class $\mathscr{C}$, then $R^q f_* \mathcal{A}^p_{X/S}$ is locally free of finite rank.

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Method

$\Rightarrow$ The hypothesis that $\mathscr{C}$ is a mild class can be falsified computationally.

The experiment: setup

Definition of the class $\mathscr{C}$

We take the class of local models of elementary Gross–Siebert type together with the local model $\{xy = t^2 + w^2, zw = tu\}$ from above.

The log de Rham complex is given by $\mathcal{A}^\bullet_{X/S} := j_*\Omega^\bullet_{U^\dagger/S^\dagger}$ over $S = \operatorname{Spec} \mathbb{C}[\![t]\!]$, and by its base change over other bases.

The family

We consider the family

$$\mathcal{X} = \{XY = W^2 + t^2V^2, \ \ ZW = tUV\} \subset \mathbb{P}^5 \times \mathbb{A}^1_t \xrightarrow{f} \mathbb{A}^1_t.$$

It has log singularities of class $\mathscr{C}$.

The experiment: result

OSCAR

Open **S**ource **C**omputer **A**lgebra **R**esearch system

written in julia; under development

[Collaborative Research Center TRR 195]

Needed capacity [under development]

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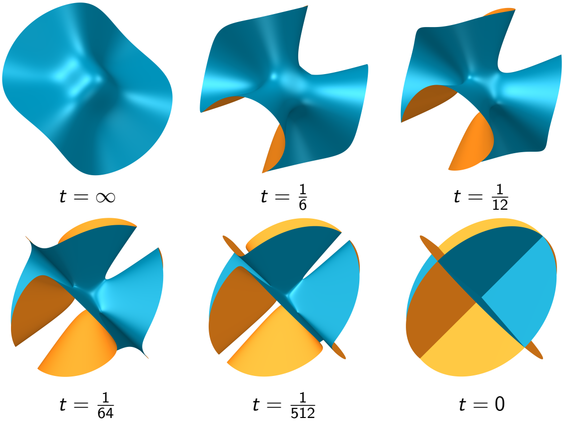
Result

$R^2 f_* \mathcal{A}^1_{\mathcal{X}/\mathbb{A}^1}$ is *not* locally free.

$\Rightarrow \mathscr{C}$ is not a mild class of log singularities.



Thank you for your attention.